Abstract

This document describes formally the Iris program logic. Every result in this document has been fully verified in Coq. The latest versions of this document and the Coq formalization can be found in the git repository at https://gitlab.mpi-sws.org/iris/iris. For further information, visit the Iris project website at http://plv.mpi-sws.org/iris/.
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1 Iris from the Ground Up

In “Iris from the Ground Up” [6], we describe Iris 3.1 in a bottom-up way. That paper is hence much more suited as an introduction to the model of Iris than this reference, which mostly contains definitions, not explanations or examples. For a list of changes in Iris since then, please consult our changelog at https://gitlab.mpi-sws.org/iris/iris/blob/master/CHANGELOG.md.
2 Algebraic Structures

2.1 OFE

The model of Iris lives in the category of Ordered Families of Equivalences (OFEs). This definition varies slightly from the original one in [2].

**Definition 1.** An ordered family of equivalences (OFE) is a tuple \((T, (\equiv_n : T \times T)_{n \in \mathbb{N}})\) satisfying

\[
\begin{align*}
\forall n. (\equiv_n) \text{ is an equivalence relation} & \quad \text{(OFE-EQUIV)} \\
\forall n, m. n \geq m \Rightarrow (\equiv_n) \subseteq (\equiv_m) & \quad \text{(OFE-MONO)} \\
\forall x, y. x = y \iff (\forall n. x \equiv_n y) & \quad \text{(OFE-LIMIT)}
\end{align*}
\]

The key intuition behind OFEs is that elements \(x\) and \(y\) are \(n\)-equivalent, notation \(x \equiv_n y\), if they are equivalent for \(n\) steps of computation, i.e., if they cannot be distinguished by a program running for no more than \(n\) steps. In other words, as \(n\) increases, \(\equiv_n\) becomes more and more refined (OFE-MONO)—and in the limit, it agrees with plain equality (OFE-LIMIT).

Notice that OFEs are just a different presentation of bisected 1-bounded ultrametric spaces, where the family of equivalence relations gives rise to the distance function (two elements that are equal for \(n\) steps are no more than \(2^{-n}\) apart).

**Definition 2.** An element \(x \in T\) of an OFE is called discrete if

\[\forall y \in T. x \equiv_0 y \Rightarrow x = y\]

An OFE \(A\) is called discrete if all its elements are discrete. For a set \(X\), we write \(\Delta X\) for the discrete OFE with \(x \equiv_n y \equiv x = x'\).

**Definition 3.** A function \(f : T \to U\) between two OFEs is non-expansive (written \(f : T \xrightarrow{n} U\)) if

\[\forall n, x, y \in T. x \equiv_n y \Rightarrow f(x) \equiv_n f(y)\]

It is contractive if

\[\forall n, x, y \in T. (\forall m < n. x \equiv_m y) \Rightarrow f(x) \equiv_n f(y)\]

Intuitively, applying a non-expansive function to some data will not suddenly introduce differences between seemingly equal data. Elements that cannot be distinguished by programs within \(n\) steps remain indistinguishable after applying \(f\).

**Definition 4.** The category OFE consists of OFEs as objects, and non-expansive functions as arrows.

Note that OFE is bicartesian closed, i.e., it has all sums, products and exponentials as well as an initial and a terminal object. In particular:

**Definition 5.** Given two OFEs \(T\) and \(U\), the set of non-expansive functions \(\{f : T \xrightarrow{n} U\}\) is itself an OFE with

\[f \equiv_n g \equiv \forall x \in T. f(x) \equiv_n g(x)\]
Definition 6. A (bi)functor \( F : \text{OFE} \to \text{OFE} \) is called locally non-expansive if its action \( F_1 \) on arrows is itself a non-expansive map. Similarly, \( F \) is called locally contractive if \( F_1 \) is a contractive map.

The function space \((-) \xrightarrow{\text{ne}} (-)\) is a locally non-expansive bifunctor. Note that the composition of non-expansive (bi)functors is non-expansive, and the composition of a non-expansive and a contractive (bi)functor is contractive.

One very important OFE is the OFE of step-indexed propositions: For every step-index, such a proposition either holds or does not hold. Moreover, if a proposition holds for some \( n \), it also has to hold for all smaller step-indices.

\[
\text{SProp} \triangleq \wp(\downarrow \mathbb{N}) = \left\{ X \in \wp(\mathbb{N}) \mid \forall n, m. n \geq m \Rightarrow n \in X \Rightarrow m \in X \right\}
\]

\[
X \xrightarrow{n} Y \triangleq \forall m \leq n. m \in X \Leftrightarrow m \in Y
\]

\[
X \sqsubseteq Y \triangleq \forall m \leq n. m \in X \Rightarrow m \in Y
\]

2.2 COFE

COFEs are complete OFEs, which means that we can take limits of arbitrary chains.

Definition 7 (Chain). Given some set \( T \) and an indexed family \( (\equiv \subseteq T \times T)_{n \in \mathbb{N}} \) of equivalence relations, a chain is a function \( c : \mathbb{N} \to T \) such that \( \forall n, m. n \leq m \Rightarrow c(m) \equiv c(n) \).

Definition 8. A complete ordered family of equivalences (COFE) is a tuple \( (\text{COFE}, \lim : \text{chain}(T) \to T) \) satisfying

\[
\forall n, c. \lim(c) \equiv c(n)
\]  

(COFE-COMPL)

Definition 9. The category COFE consists of COFEs as objects, and non-expansive functions as arrows.

The function space \( T \xrightarrow{\text{ne}} U \) is a COFE if \( U \) is a COFE (i.e., the domain \( T \) can actually be just an OFE). SProp as defined above is complete, i.e., it is a COFE.

Completeness is necessary to take fixed-points.

Theorem 1 (Banach’s fixed-point). Given an inhabited COFE \( T \) and a contractive function \( f : T \to T \), there exists a unique fixed-point \( \text{fix}_T f \) such that \( f(\text{fix}_T f) = \text{fix}_T f \). Moreover, this theorem also holds if \( f \) is just non-expansive, and \( f^k \) is contractive for an arbitrary \( k \).

Theorem 2 (America and Rutten [1, 3]). Let \( 1 \) be the discrete COFE on the unit type: \( 1 \triangleq \Delta\{()\} \). Given a locally contractive bifunctor \( G : \text{COFE}^{\text{op}} \times \text{COFE} \to \text{COFE} \), and provided that \( G(1, 1) \) is inhabited, then there exists a unique\(^1\) \( \text{COFE} T \) such that \( G(T^{\text{op}}, T) \cong T \) (i.e., the two are isomorphic in \( \text{COFE} \)).

\(^1\)Uniqueness is not proven in Coq.
2.3 RA

Definition 10. A resource algebra (RA) is a tuple 
\((M, \nabla : M \rightarrow Prop, |−| : M \rightarrow M^2, (\cdot) : M \times M \rightarrow M)\) satisfying:

\[
\begin{align*}
\forall a, b, c. (a \cdot b) \cdot c &= a \cdot (b \cdot c) \\
\forall a, b. a \cdot b &= b \cdot a \\
\forall a. |a| \in M \Rightarrow |a| \cdot a &= a \\
\forall a. |a| \in M \Rightarrow ||a|| = |a| \\
\forall a, b. |a| \in M \wedge a \preceq b \Rightarrow |b| \in M \wedge |a| \preceq |b| \\
\forall a, b. \nabla(a \cdot b) \Rightarrow \nabla(a)
\end{align*}
\]

where \(M^2 \triangleq M \uplus \{\bot\}\) \(a^? \cdot \bot \triangleq \bot \cdot a^? \triangleq a^?\)

\[
\begin{align*}
a \preceq b \triangleq \exists c \in M. b = a \cdot c
\end{align*}
\]

Here, Prop is the set of (meta-level) propositions. Think of Prop in Coq or \(B\) in classical mathematics.

RAs are closely related to Partial Commutative Monoids (PCMs), with two key differences:

1. The composition operation on RAs is total (as opposed to the partial composition operation of a PCM), but there is a specific subset of valid elements that is compatible with the composition operation (RA-valid-op). These valid elements are identified by the validity predicate \(\nabla\).

   This take on partiality is necessary when defining the structure of higher-order ghost state, cameras, in the next subsection.

2. Instead of a single unit that is an identity to every element, we allow for an arbitrary number of units, via a function \(|−|\) assigning to an element \(a\) its (duplicable) core \(|a|\), as demanded by RA-core-id. We further demand that \(|−|\) is idempotent (RA-core-idem) and monotone (RA-core-mono) with respect to the extension order, defined similarly to that for PCMs (RA-incl).

   Notice that the domain of the core is \(M^2\), a set that adds a dummy element \(\bot\) to \(M\). Thus, the core can be partial: not all elements need to have a unit. We use the metavariable \(a^?\) to indicate elements of \(M^2\). We also lift the composition \((\cdot)\) to \(M^2\). Partial cores help us to build interesting composite RAs from smaller primitives.

   Notice also that the core of an RA is a strict generalization of the unit that any PCM must provide, since \(|−|\) can always be picked as a constant function.

Definition 11. It is possible to do a frame-preserving update from \(a \in M\) to \(B \subseteq M\), written \(a \rightsquigarrow B\), if

\[
\forall a^? \in M^2, \nabla(a \cdot a^?) \Rightarrow \exists b \in B. \nabla(b \cdot a^?)
\]

We further define \(a \rightsquigarrow b \triangleq a \rightsquigarrow \{b\}\).

The proposition \(a \rightsquigarrow B\) says that every element \(a^?\) compatible with \(a\) (we also call such elements frames), must also be compatible with some \(b \in B\). Notice that \(a^?\) could be \(\bot\), so the frame-preserving update can also be applied to elements that have no frame. Intuitively, this means that whatever assumptions the rest of the program is making about the state of \(\gamma\), if these assumptions
are compatible with $a$, then updating to $b$ will not invalidate any of these assumptions. Since Iris ensures that the global ghost state is valid, this means that we can soundly update the ghost state from $a$ to a non-deterministically picked $b \in B$.

### 2.4 Cameras

**Definition 12.** A camera is a tuple $(M : \text{OFE}, \mathcal{V} : M \xrightarrow{\text{nc}} \text{SProp}, |- : M \xrightarrow{\text{nc}} M^2, (\cdot) : M \times M \xrightarrow{\text{nc}} M)$ satisfying:

- $\forall a, b, c. (a \cdot b) \cdot c = a \cdot (b \cdot c)$ (camera-assoc)
- $\forall a, b. a \cdot b = b \cdot a$ (camera-comm)
- $\forall a. |a| \in M \Rightarrow |a| \cdot a = a$ (camera-core-id)
- $\forall a. |a| \in M \Rightarrow ||a|| = |a|$ (camera-core-iden)
- $\forall a, b. |a| \in M \land a \triangleleft b \Rightarrow |b| \in M \land |a| \triangleleft |b|$ (camera-core-mono)
- $\forall a, b. \mathcal{V}(a \cdot b) \subseteq \mathcal{V}(a)$ (camera-valid-op)
- $\forall n, a, b_1, b_2. n \in \mathcal{V}(a) \land a \triangleleft b_1 \cdot b_2 \Rightarrow \exists c_1, c_2. a = c_1 \cdot c_2 \land c_1^n = b_1 \land c_2^n = b_2$ (camera-extend)

where

- $a \triangleleft b \triangleq \exists c. b = a \cdot c$ (camera-incl)
- $a \triangleleft b \triangleq \exists c. b = a \cdot c$ (camera-inclN)

This is a natural generalization of RAs over OFEs\(^2\). All operations have to be non-expansive, and the validity predicate $\mathcal{V}$ can now also depend on the step-index. We define the plain $\mathcal{V}$ as the “limit” of the step-indexed approximation:

$$\mathcal{V}(a) \triangleq \forall n. n \in \mathcal{V}(a)$$

The **extension axiom** (camera-extend). Notice that the existential quantification in this axiom is constructive, i.e., it is a sigma type in Coq. The purpose of this axiom is to compute $a_1, a_2$ completing the following square:

$$
\begin{array}{ccc}
  a & \equiv & b \\
  || & \equiv & || \\
  a_1 \cdot a_2 & \equiv & b_1 \cdot b_2 \\
\end{array}
$$

\(^2\)The reader may wonder why on earth we call them “cameras”. The reason, which may not be entirely convincing, is that “camera” was originally just used as a comfortable pronunciation of “CMRA”, the name used in earlier Iris papers. CMRA was originally supposed to be an acronym for “complete metric resource algebras” (or something like that), but we were never very satisfied with it and thus ended up never spelling it out. To make matters worse, the “complete” part of CMRA is now downright misleading, for whereas previously the carrier of a CMRA was required to be a COFE (complete OFE), we have relaxed that restriction and permit it to be an (incomplete) OFE. For these reasons, we have decided to stick with the name “camera”, for purposes of continuity, but to drop any pretense that it stands for something.
where the $n$-equivalence at the bottom is meant to apply to the pairs of elements, i.e., we demand $a_1 \equiv b_1$ and $a_2 \equiv b_2$. In other words, extension carries the decomposition of $b$ into $b_1$ and $b_2$ over the $n$-equivalence of $a$ and $b$, and yields a corresponding decomposition of $a$ into $a_1$ and $a_2$. This operation is needed to prove that $\triangleright$ commutes with separating conjunction:

\[ \triangleright(P \cdot Q) \Leftrightarrow \triangleright P \cdot \triangleright Q \]

**Definition 13.** An element $\varepsilon$ of a camera $M$ is called the unit of $M$ if it satisfies the following conditions:

1. $\varepsilon$ is valid:
   \[ \forall n. n \in V(\varepsilon) \]
2. $\varepsilon$ is a left-identity of the operation:
   \[ \forall a \in M. \varepsilon \cdot a = a \]
3. $\varepsilon$ is its own core:
   \[ |\varepsilon| = \varepsilon \]

**Lemma 1.** If $M$ has a unit $\varepsilon$, then the core $|\cdot|$ is total, i.e., $\forall a. |a| \in M$.

**Definition 14.** It is possible to do a frame-preserving update from $a \in M$ to $B \subseteq M$, written $a \mp B$, if

\[ \forall n, a_i. n \in V(a \cdot a_i) \Rightarrow \exists b \in B. n \in V(b \cdot a_i) \]

We further define $a \mp B \equiv a \mp \{b\}$.

Note that for RAs, this and the RA-based definition of a frame-preserving update coincide.

**Definition 15.** A camera $M$ is discrete if it satisfies the following conditions:

1. $M$ is a discrete COFE
2. $V$ ignores the step-index:
   \[ \forall a \in M. 0 \in V(a) \Rightarrow \forall n. n \in V(a) \]

Note that every RA is a discrete camera, by picking the discrete COFE for the equivalence relation. Furthermore, discrete cameras can be turned into RAs by ignoring their COFE structure, as well as the step-index of $V$.

**Definition 16 (Camera homomorphism).** A function $f : M_1 \rightarrow M_2$ between two cameras is a camera homomorphism if it satisfies the following conditions:

1. $f$ is non-expansive
2. $f$ commutes with composition:
   \[ \forall a_1 \in M_1, a_2 \in M_1. f(a_1) \cdot f(a_2) = f(a_1 \cdot a_2) \]
3. $f$ commutes with the core:
   \[ \forall a \in M_1. |f(a)| = f(|a|) \]
4. $f$ preserves validity:
   \[ \forall n, a \in M_1. n \in V(a) \Rightarrow n \in V(f(a)) \]

**Definition 17.** The category Camera consists of cameras as objects, and camera homomorphisms as arrows.

Note that every object/arrow in Camera is also an object/arrow of OFE. The notion of a locally non-expansive (or contractive) bifunctor naturally generalizes to bifunctors between these categories.
3 OFE and COFE Constructions

3.1 Trivial Pointwise Lifting

The (C)OFE structure on many types can be easily obtained by pointwise lifting of the structure of the components. This is what we do for option $T^?$, product $(M_i)_{i \in I}$ (with $I$ some finite index set), sum $T + T'$ and finite partial functions $K \xrightarrow{\text{fin}} M$ (with $K$ infinite countable).

3.2 Next (Type-Level Later)

Given a OFE $T$, we define $\triangleright T$ as follows (using a datatype-like notation to define the type):

$$\triangleright T \triangleq \text{next}(x : T)$$

$$\text{next}(x)^n \triangleq \text{next}(y) \triangleq n = 0 \lor x^{n-1} = y$$

Note that in the definition of the carrier $\triangleright T$, next is a constructor (like the constructors in Coq), i.e., this is short for $\{\text{next}(x) \mid x \in T\}$.

$\triangleright(-)$ is a locally contractive functor from OFE to OFE.

3.3 Uniform Predicates

Given a camera $M$, we define the COFE $\text{UPred}(M)$ of uniform predicates over $M$ as follows:

$$\text{UPred}(M) \triangleq M \xrightarrow{\text{mon,ne}} \text{SProp}$$

$$\Phi \equiv \Psi \triangleq \forall m, a. m \in \mathcal{V}(a) \Rightarrow (m \in \Phi(a) \iff m \in \Psi(a))$$

$$\Phi^n \equiv \Psi^n \triangleq \forall m \leq n, a. m \in \mathcal{V}(a) \Rightarrow (m \in \Phi(a) \iff m \in \Psi(a))$$

You can think of uniform predicates as monotone, step-indexed predicates over a camera that “ignore” invalid elements (as defined by the quotient).

$\text{UPred}(-)$ is a locally non-expansive functor from Camera to COFE.

It is worth noting that the above quotient admits canonical representatives. More precisely, one can show that every equivalence class contains exactly one element $P_0$ such that:

$$\forall n, a. (\mathcal{V}(a)^n \subseteq P_0(a)) \Rightarrow n \in P_0(a)$$

(UPred-canonical)

Intuitively, this says that $P_0$ trivially holds whenever the resource is invalid. Starting from any element $P$, one can find this canonical representative by choosing $P_0(a) := \{n \mid n \in \mathcal{V}(a) \Rightarrow n \in P(a)\}$.

Hence, as an alternative definition of $\text{UPred}$, we could use the set of canonical representatives. This alternative definition would save us from using a quotient. However, the definitions of the various connectives would get more complicated, because we have to make sure they all verify UPred-canonical, which sometimes requires some adjustments. We would moreover need to prove one more property for every logical connective.
4 RA and Camera Constructions

4.1 Product
Given a family \((M_i)_{i \in I}\) of cameras \((I \text{ finite})\), we construct a camera for the product \(\prod_{i \in I} M_i\) by lifting everything pointwise.

Frame-preserving updates on the \(M_i\) lift to the product:

\[
\text{prod-update} \quad a \leadsto_{M_i} B \\
\{f[i \leftarrow a] \leadsto \{f[i \leftarrow b] \mid b \in B\}
\]

4.2 Sum
The sum camera \(M_1 + \ominus M_2\) for any cameras \(M_1\) and \(M_2\) is defined as (again, we use a datatype-like notation):

\[
M_1 + \ominus M_2 \triangleq \text{inl}(a_1 : M_1) \mid \text{inr}(a_2 : M_2) \mid \_ \\
\mathcal{V}(\_) \triangleq \emptyset \\
\mathcal{V}(\text{inl}(a)) \triangleq \mathcal{V}_1(a) \\
\text{inl}(a_1) \cdot \text{inl}(b_1) \triangleq \text{inl}(a_1 \cdot b_1) \\
|\text{inl}(a_1)| \triangleq \begin{cases} 
\bot & \text{if } |a_1| = \bot \\
|\text{inl}(|a_1|)| & \text{otherwise}
\end{cases}
\]

Above, \(\mathcal{V}_1\) refers to the validity of \(M_1\). The validity, composition and core for \(\text{inr}\) are defined symmetrically. The remaining cases of the composition and core are all \(\ominus\).

Notice that we added the artificial “invalid” (or “undefined”) element \(\_\) to this camera just in order to make certain compositions of elements (in this case, \(\text{inl}\) and \(\text{inr}\)) invalid.

The step-indexed equivalence is inductively defined as follows:

\[
\begin{align*}
x \equiv y & \quad \text{inl}(x) \equiv \text{inl}(y) \\
x \equiv y & \quad \text{inr}(x) \equiv \text{inr}(y) \\
\_ \equiv \_ & \quad x \equiv y
\end{align*}
\]

We obtain the following frame-preserving updates, as well as their symmetric counterparts:

\[
\text{sum-update} \quad a \leadsto_{M_i} B \\
\text{sum-swap} \quad \forall a_\_ \in M, n. n \notin \mathcal{V}(a \cdot a_\_) \\
\text{sum-swap} \quad \text{inl}(a) \leadsto \text{inr}(b)
\]

Crucially, the second rule allows us to swap the “side” of the sum that the camera is on if \(\mathcal{V}\) has no possible frame.

4.3 Option
The definition of the camera/RA axioms already lifted the composition operation on \(M\) to one on \(M^\_\). We can easily extend this to a full camera by defining a suitable core, namely

\[
\begin{align*}
\bot & \triangleq \bot \\
|a^\_| & \triangleq |a| & \text{If } a^\_ \neq \bot
\end{align*}
\]
Notice that this core is total, as the result always lies in $M^?$ (rather than in $M^??$).

### 4.4 Finite Partial Functions

Given some infinite countable $K$ and some camera $M$, the set of finite partial functions $K^\text{fin} \rightarrow M$ is equipped with a camera structure by lifting everything pointwise.

We obtain the following frame-preserving updates:

- **FPFN-ALLOC-STRONG**
  \[
  \emptyset \rightsquigarrow \{ [\gamma \leftarrow a] \mid \gamma \in G \}
  \]

- **FPFN-ALLOC**
  \[
  \emptyset \rightsquigarrow \{ [\gamma \leftarrow a] \mid \gamma \in K \}
  \]

- **FPFN-UPDATE**
  \[
  a \mapsto_M B 
  \]
  \[
  f \left[ i \leftarrow a \right] \rightsquigarrow \{ f \left[ i \leftarrow b \right] \mid b \in B \}
  \]

Above, $\mathcal{V}$ refers to the (full) validity of $M$.

$K^\text{fin} \rightarrow -$ is a locally non-expansive functor from **Camera** to **Camera**.

### 4.5 Agreement

Given some OFE $T$, we define the camera $\text{Ag}(T)$ as follows:

\[
\begin{align*}
\text{Ag}(T) & \triangleq \{ a \in \wp^\text{fin}(T) \mid a \neq \emptyset \} / \sim \\
& a \equiv b \triangleq (\forall x \in a. \exists y \in b. x \equiv b) \land (\forall y \in b. \exists x \in a. x \equiv y)
\end{align*}
\]

where $a \sim b \triangleq \forall n. a \equiv n b$

\[
\begin{align*}
\mathcal{V}(a) & \triangleq \{ n \mid \forall x, y \in a. x \equiv n y \} \\
|a| & \equiv a \\
a \cdot b & \equiv a \cup b
\end{align*}
\]

$\text{Ag}(-)$ is a locally non-expansive functor from **OFE** to **Camera**.

We define a non-expansive injection $\text{ag}$ into $\text{Ag}(T)$ as follows:

\[
\text{ag}(x) \triangleq \{ x \}
\]

There are no interesting frame-preserving updates for $\text{Ag}(T)$, but we can show the following:

- **AG-VAL**
  \[
  \mathcal{V}(\text{ag}(x))
  \]

- **AG-DUP**
  \[
  \text{ag}(x) = \text{ag}(x) \cdot \text{ag}(x)
  \]

- **AG-AGREE**
  \[
  n \in \mathcal{V}(\text{ag}(x) \cdot \text{ag}(y)) \Rightarrow x \equiv n y
  \]

### 4.6 Exclusive Camera

Given an OFE $T$, we define a camera $\text{Ex}(T)$ such that at most one $x \in T$ can be owned:

\[
\begin{align*}
\text{Ex}(T) & \triangleq \text{ex}(T) \mid \dagger \\
\mathcal{V}(a) & \triangleq \{ n \mid a \equiv n \dagger \}
\end{align*}
\]
All cases of composition go to \( \bot \).

\[
|\text{ex}(x)| \triangleq \bot \\
|\bot| \triangleq \bot
\]

Remember that \( \bot \) is the “dummy” element in \( M \) indicating (in this case) that \( \text{ex}(x) \) has no core.

The step-indexed equivalence is inductively defined as follows:

\[
\frac{x \equiv y}{\text{ex}(x) \equiv \text{ex}(y)} \\
\frac{\bot \equiv \bot}{\bot \equiv \bot}
\]

\( \text{Ex}(\_\_\_) \) is a locally non-expansive functor from \( \text{OFE} \) to \( \text{Camera} \).

We obtain the following frame-preserving update:

\[
\text{ex-update} \\
\text{ex}(x) \leadsto \text{ex}(y)
\]

### 4.7 Fractions

We define an RA structure on the rational numbers in \((0, 1]\) as follows:

\[
\text{Frac} \triangleq \frac{\text{frac}(\mathbb{Q} \cap (0, 1])}{\bot} \\
\text{V}(a) \triangleq a \neq \bot \\
\text{frac}(q_1) \cdot \text{frac}(q_2) \triangleq \text{frac}(q_1 + q_2) \quad \text{if } q_1 + q_2 \leq 1 \\
|\text{frac}(x)| \triangleq \bot \\
|\bot| \triangleq \bot
\]

All remaining cases of composition go to \( \bot \). Frequently, we will write just \( x \) instead of \( \text{frac}(x) \).

The most important property of this RA is that \( 1 \) has no frame. This is useful in combination with \( \text{sum-swap} \), and also when used with pairs:

\[
\text{pair-frac-change} \\
(1, a) \leadsto (1, b)
\]

### 4.8 Authoritative

Given a camera \( M \), we construct \( \text{Auth}(M) \) modeling someone owning an \textit{authoritative} element \( a \) of \( M \), and others potentially owning fragments \( b \preceq a \) of \( a \). We assume that \( M \) has a unit \( \varepsilon \), and hence its core is total. (If \( M \) is an exclusive monoid, the construction is very similar to a half-ownership monoid with two asymmetric halves.)

\[
\text{Auth}(M) \triangleq \text{Ex}(M)^2 \times M \\
\text{V}((x, b)) \triangleq \{ n \mid (x = \bot \land n \in \text{V}(b)) \lor (\exists a. x = \text{ex}(a) \land b \preceq_a a \land n \in \text{V}(a)) \} \\
(x_1, b_1) \cdot (x_2, b_2) \triangleq (x_1 \cdot x_2, b_2 \cdot b_2) \\
|(x, b)| \triangleq (\bot, |b|) \\
(x_1, b_1) \overset{n}{\Rightarrow} (x_2, b_2) \triangleq x_1 \overset{n}{=} x_2 \land b_1 \overset{n}{=} b_2
\]
Note that \((\bot, \varepsilon)\) is the unit and asserts no ownership whatsoever, but \((\text{ex}(\varepsilon), \varepsilon)\) asserts that the authoritative element is \(\varepsilon\).

Let \(a, b \in M\). We write \(\bullet a\) for full ownership \((\text{ex}(a), \varepsilon)\) and \(\circ b\) for fragmental ownership \((\bot, b)\) and \(\bullet a, \circ b\) for combined ownership \((\text{ex}(a), b)\).

The frame-preserving update involves the notion of a local update:

**Definition 18.** It is possible to do a local update from \(a_1\) and \(b_1\) to \(a_2\) and \(b_2\), written \((a_1, b_1) \xrightarrow{1} (a_2, b_2)\), if
\[
\forall n, a_i^2, n \in \mathcal{V}(a_1) \land a_1^2 = b_1 \cdot a_i^2 \Rightarrow n \in \mathcal{V}(a_2) \land a_2^2 = b_2 \cdot a_i^2
\]

In other words, the idea is that for every possible frame \(a_i^2\) completing \(b_1\) to \(a_1\), the same frame also completes \(b_2\) to \(a_2\).

We then obtain
\[
\text{auth-update}
\]
\[
\begin{align*}
(a_1, b_1) & \xrightarrow{1} (a_2, b_2) \\
\bullet a_1, \circ b_1 & \xrightarrow{1} \bullet a_2, \circ b_2
\end{align*}
\]

**4.9 STS with Tokens**

Given a state-transition system (STS, i.e., a directed graph) \((S, \to \subseteq S \times S)\), a set of tokens \(T\), and a labeling \(L : S \to \wp(T)\) of protocol-owned tokens for each state, we construct an RA modeling an authoritative current state and permitting transitions given a bound on the current state and a set of locally-owned tokens.

The construction follows the idea of STSs as described in CaReSL [8]. We first lift the transition relation to \(S \times \wp(T)\) (implementing a law of token conservation) and define a stepping relation for the frame of a given token set:
\[
(s, T) \to (s', T') \triangleq s \to s' \land L(s) \cup T = L(s') \cup T'
\]
\[
s \xrightarrow{T} s' \triangleq \exists T_1, T_2, T_1 \# L(s) \cup T \land (s, T_1) \to (s', T_2)
\]

We further define closed sets of states (given a particular set of tokens) as well as the closure of a set:
\[
\text{closed}(S, T) \triangleq \forall s \in S, L(s) \# T \land \left(\forall s', s \xrightarrow{T} s' \Rightarrow s' \in S\right)
\]
\[
\uparrow(S, T) \triangleq \left\{s' \in S \mid \exists s \in S, s \xrightarrow{T} s'\right\}
\]

The STS RA is defined as follows
\[
M \triangleq \text{auth}(s : S, T : \wp(T) \mid L(s) \# T) \mid \text{frag}(S : \wp(S), T : \wp(T) \mid \text{closed}(S, T) \land S \neq \emptyset) \mid \frac{\dagger}{\dagger}
\]
\[
\nabla(a) \triangleq a \neq \frac{\dagger}{\dagger}
\]
\[
\text{frag}(S_1, T_1) \cdot \text{frag}(S_2, T_2) \triangleq \text{frag}(S_1 \cap S_2, T_1 \cup T_2) \quad \text{if } T_1 \# T_2 \text{ and } S_1 \cap S_2 \neq \emptyset
\]
\[
\text{frag}(S, T) \cdot \text{auth}(s, T') \triangleq \text{auth}(s, T') \cdot \text{frag}(S, T) \triangleq \text{auth}(s, T \cup T') \quad \text{if } T \# T' \text{ and } s \in S
\]
\[
\left|\text{frag}(S, T)\right| \triangleq \text{frag}(\uparrow(S, \emptyset), \emptyset)
\]
\[
\left|\text{auth}(s, T)\right| \triangleq \text{frag}(\uparrow\{s\}, \emptyset, \emptyset)
\]

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The remaining cases are all \( \Rightarrow \).

We will need the following frame-preserving update:

\[
\begin{align*}
\text{sts-step} & \quad (s, T) \rightarrow^* (s', T') \\
\text{auth} & \quad \text{auth}(s, T) \leadsto \text{auth}(s', T')
\end{align*}
\]

\[
\begin{align*}
\text{sts-weaken} & \quad \text{closed}(S_2, T_2) \\
\text{frag} & \quad \text{frag}(S_1, T_1) \leadsto \text{frag}(S_2, T_2)
\end{align*}
\]

**The core is not a homomorphism.** The core of the STS construction is only satisfying the RA axioms because we are not demanding the core to be a homomorphism—all we demand is for the core to be monotone with respect the RA-INCL.

In other words, the following does not hold for the STS core as defined above:

\[ |a| \cdot |b| = |a \cdot b| \]

To see why, consider the following STS:

Now consider the following two elements of the STS RA:

\[
a \triangleq \text{frag}(\{s_1, s_2\}, \{T_1\}) \quad b \triangleq \text{frag}(\{s_1, s_3\}, \{T_2\})
\]

We have:

\[
|a| = \text{frag}(\{s_1, s_2, s_4\}, \emptyset) \quad |b| = \text{frag}(\{s_1, s_3, s_4\}, \emptyset)
\]

\[
|a| \cdot |b| = \text{frag}(\{s_1, s_4\}, \emptyset) \neq |a \cdot b| = \text{frag}(\{s_1\}, \emptyset)
\]
5 Base Logic

The base logic is parameterized by an arbitrary camera $M$ having a unit $\varepsilon$. By Lemma 1, this means that the core of $M$ is a total function, so we will treat it as such in the following. This defines the structure of resources that can be owned.

As usual for higher-order logics, you can furthermore pick a signature $S = (T, F, A)$ to add more types, symbols and axioms to the language. You have to make sure that $T$ includes the base types:

$$T \supseteq \{M, i\text{Prop}\}$$

Elements of $T$ are ranged over by $T$.

Each function symbol in $F$ has an associated arity comprising a natural number $n$ and an ordered list of $n + 1$ types $\tau$ (the grammar of $\tau$ is defined below, and depends only on $T$). We write $F : \tau_1, \ldots, \tau_n \rightarrow \tau_{n+1} \in F$ to express that $F$ is a function symbol with the indicated arity.

Furthermore, $A$ is a set of axioms, that is, terms $t$ of type $i\text{Prop}$. Again, the grammar of terms and their typing rules are defined below, and depends only on $T$ and $F$, not on $A$. Elements of $A$ are ranged over by $A$.

5.1 Grammar

Syntax. Iris syntax is built up from a signature $S$ and a countably infinite set $\text{Var}$ of variables (ranged over by metavariables $x$, $y$, $z$). Below, $a$ ranges over $M$ and $i$ ranges over $\{1, 2\}$.

$$\tau ::= T \mid 0 \mid 1 \mid \tau + \tau \mid \tau \times \tau \mid \tau \rightarrow \tau$$

$$t, P, \Phi ::= x \mid F(t_1, \ldots, t_n) \mid \text{abort} t \mid () \mid (t, t) \mid \pi_i t \mid \lambda x : \tau. t \mid t(t) \mid$$

$$\text{inj}_i t \mid \text{match} t \mid \text{match} t \text{ with inj}_1 x. t \mid \text{inj}_2 x. t \text{ end} \mid a \mid | t | \mid t \cdot t$$

$$\text{False} \mid \text{True} \mid t = \tau t \mid P \Rightarrow P \mid P \land P \mid P \lor P \mid P * P \mid P - * P$$

$$\mu x : \tau. t \mid \exists x : \tau. P \mid \forall x : \tau. P \mid \text{Own}(t) \mid \text{V}(t) \mid \Box P \mid \text{□} P \mid \triangleright P \mid \triangleright P$$

Well-typedness forces recursive definitions to be guarded: In $\mu x. t$, the variable $x$ can only appear under the later $\triangleright$ modality. Furthermore, the type of the definition must be complete. The type $i\text{Prop}$ is complete, and if $\tau$ is complete, then so is $\tau' \rightarrow \tau$.

Note that the modalities $\triangleright, \Box, \text{□}$ and $\triangleright$ bind more tightly than $\ast, \rightarrow, \land, \lor$, and $\Rightarrow$.

Variable conventions. We assume that, if a term occurs multiple times in a rule, its free variables are exactly those binders which are available at every occurrence.

5.2 Types

Iris terms are simply-typed. The judgment $\Gamma \vdash t : \tau$ expresses that, in variable context $\Gamma$, the term $t$ has type $\tau$.

A variable context, $\Gamma = x_1 : \tau_1, \ldots, x_n : \tau_n$, declares a list of variables and their types. In writing $\Gamma, x : \tau$, we presuppose that $x$ is not already declared in $\Gamma$. 

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Well-typed terms

\[ \frac{x : \tau \vdash x : \tau}{\Gamma \vdash t : \tau} \quad \frac{\Gamma, x : \tau' \vdash t : \tau}{\Gamma, x : \tau' \vdash t : \tau} \quad \frac{\Gamma, x : \tau', y : \tau' \vdash t : \tau}{\Gamma, x : \tau', y : \tau' \vdash t : \tau} \quad \frac{\Gamma, x : \tau', y : \tau', \Gamma_2 \vdash t : \tau}{\Gamma, x : \tau', y : \tau', \Gamma_2 \vdash t[y/x, x/y] : \tau} \]

\[ \frac{\Gamma \vdash t_1 : \tau_1 \quad \cdots \quad \Gamma \vdash t_n : \tau_n}{\Gamma \vdash F(t_1, \ldots, t_n) : \tau_{n+1}} \quad \frac{F : \tau_1, \ldots, \tau_n \rightarrow \tau_{n+1} \in \mathcal{F}}{\Gamma \vdash t : 0} \quad \frac{\Gamma \vdash \text{abort} t : \tau}{\Gamma \vdash (\cdot) : 1} \]

\[ \frac{\Gamma \vdash t : \tau_1}{\Gamma \vdash (t, u) : \tau_1 \times \tau_2} \quad \frac{\Gamma \vdash t : \tau_1 \times \tau_2}{\Gamma \vdash \pi_i t : \tau_i} \quad \frac{\Gamma \vdash t : \tau_1 + \tau_2}{\Gamma \vdash \text{match } t \text{ with } \text{inj}_1 x.t_1 | \text{inj}_2 y.t_2 \text{ end } : \tau} \]

\[ \frac{\Gamma \vdash \tau \rightarrow \tau'}{\Gamma \vdash t(u) : \tau'} \quad \frac{\Gamma \vdash a : M}{\Gamma \vdash [a] : M} \quad \frac{\Gamma \vdash a : M}{\Gamma \vdash a \cdot b : M} \]

\[ \frac{\Gamma \vdash \text{False} : \text{iProp}}{\Gamma \vdash \text{True} : \text{iProp}} \]

\[ \frac{\Gamma \vdash \text{False} : \text{iProp}}{\Gamma \vdash \text{True} : \text{iProp}} \quad \frac{\Gamma \vdash P : \text{iProp} \quad \Gamma \vdash Q : \text{iProp}}{\Gamma \vdash P \land Q : \text{iProp}} \quad \frac{\Gamma \vdash P : \text{iProp} \quad \Gamma \vdash Q : \text{iProp}}{\Gamma \vdash P \lor Q : \text{iProp}} \]

\[ \frac{\Gamma \vdash \text{match } t \text{ with } \text{inj}_1 x.t_1 | \text{inj}_2 y.t_2 \text{ end } : \tau}{\Gamma \vdash \tau \rightarrow \tau'} \quad \frac{\Gamma \vdash \tau \rightarrow \tau'}{\Gamma \vdash \tau} \]

\[ \frac{\Gamma, x : \tau \vdash t : \tau}{\Gamma \vdash \mu x : \tau.t : \tau} \quad \frac{x \text{ is guarded in } t \quad \tau \text{ is complete and inhabited}}{\Gamma \vdash \exists x : \tau.P : \text{iProp}} \]

\[ \frac{\Gamma \vdash \forall x : \tau.P : \text{iProp}}{\Gamma \vdash \text{Own } (a) : \text{iProp}} \quad \frac{\Gamma \vdash a : \tau \quad \tau \text{ is a camera}}{\Gamma \vdash \forall (a) : \text{iProp}} \quad \frac{\Gamma \vdash P : \text{iProp}}{\Gamma \vdash \Box P : \text{iProp}} \]

\[ \frac{\Gamma \vdash P : \text{iProp}}{\Gamma \vdash \Box P : \text{iProp}} \quad \frac{\Gamma \vdash P : \text{iProp}}{\Gamma \vdash \triangleright P : \text{iProp}} \quad \frac{\Gamma \vdash P : \text{iProp}}{\Gamma \vdash \triangleright P : \text{iProp}} \]

5.3 Proof Rules

The judgment \( \Gamma \vdash P : Q \) says that with free variables \( \Gamma \), proposition \( Q \) holds whenever assumption \( P \) holds. Most of the rules will entirely omit the variable contexts \( \Gamma \). In this case, we assume the same arbitrary context is used for every constituent of the rules. Axioms \( \Gamma \vdash P : Q \) indicate that both \( \Gamma \vdash P : Q \) and \( \Gamma \vdash Q : P \) are proof rules of the logic.

\[ \Gamma \vdash P : Q \]
Laws of intuitionistic higher-order logic with equality. This is entirely standard.

\[
\begin{array}{|c|c|c|}
\hline
\text{ASM} & P \vdash P & \text{Cut} \quad P \vdash Q, Q \vdash R \\
\text{Refl.} & \text{True} \vdash t =_\tau t & \text{Eq} \quad \Gamma, x : \tau \vdash \text{iProp} \\
& \Gamma \mid P \vdash Q[t/x] & \Gamma \mid P \vdash t =_\tau t' \\
\hline
\end{array}
\]

Furthermore, we have the usual \(\eta\) and \(\beta\) laws for projections, \(\text{abort}\), sum elimination, \(\lambda\) and \(\mu\).

Laws of (affine) bunched implications.

\[
\begin{array}{|c|c|c|}
\hline
\text{VIL} & P \vdash Q & \text{VIR} \quad P \vdash R \\
\text{\forall E} & \forall E \quad P \vdash R & Q \vdash R \\
\text{\Rightarrow I} & \Rightarrow I \quad P \wedge Q \vdash R & P \vdash R \\
\text{\Rightarrow E} & \Rightarrow E \\
\hline
\end{array}
\]

Laws for the plainness modality.

\[
\begin{array}{|c|c|c|}
\hline
\text{\blacksquare-Mono} & P \vdash Q \quad \text{\blacksquare-E} & (\blacksquare P \Rightarrow \blacksquare Q) \vdash \blacksquare(\blacksquare P \Rightarrow Q) \\
\text{\blacksquare-P} & \Box P \vdash \Box Q & \Box((P \Rightarrow Q) \wedge (Q \Rightarrow P)) \vdash P \Rightarrow Q \\
\hline
\end{array}
\]

Laws for the persistence modality.

\[
\begin{array}{|c|c|c|}
\hline
\text{\Box-Mono} & P \vdash Q & \text{\Box-E} \quad (\Box P \Rightarrow \Box Q) \vdash \Box(\Box P \Rightarrow Q) \\
\text{\Box-P} & \Box P \vdash \Box P & \forall x. \Box P \vdash \Box \forall x. P \\
\Box \exists x. P \vdash \exists x. \Box P \\
\hline
\end{array}
\]

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Laws for the later modality.

\[
\begin{align*}
P & \vdash Q \\
\Rightarrow P & \vdash \Rightarrow Q \\
P & \vdash P \\
\Rightarrow \exists x. P & \vdash \Rightarrow (\exists x. \Rightarrow P) \\
\Rightarrow P & \vdash \Rightarrow False \lor (\Rightarrow False \Rightarrow P) \\
\Rightarrow (P \land Q) & \vdash \Rightarrow (P \land \Rightarrow Q) \\
\ framed & \vdash \Rightarrow P \\
\end{align*}
\]

Laws for resources and validity.

\[
\begin{align*}
\text{Own}(a) \ast \text{Own}(b) & \vdash \Rightarrow \text{Own}(a \cdot b) \\
\text{Own}(a) & \vdash \Box \text{Own}(\{a\}) \\
\text{True} & \vdash \text{Own}(\varepsilon) \\
\Rightarrow \text{Own}(a) & \vdash \exists b. \text{Own}(b) \land \Rightarrow (a = b) \\
\text{Own}(a) & \vdash \forall (\text{Own}(a)) \\
\forall (a \cdot b) & \vdash \forall (\text{Own}(a)) \\
\forall (a) & \vdash \Box \forall (\text{Own}(a)) \\
\end{align*}
\]

Laws for the basic update modality.

\[
\begin{align*}
P & \vdash Q \\
\Rightarrow P & \vdash \Rightarrow Q \\
P & \vdash \Rightarrow P \\
\Rightarrow \exists b. \text{Own}(b) & \vdash \Rightarrow (Q \ast P) \\
\end{align*}
\]

The premise in \textsc{upd-update} is a \textit{meta-level} side-condition that has to be proven about \textit{a} and \textit{B}.

### 5.4 Consistency

The consistency statement of the logic reads as follows: For any \(n\), we have

\[\neg(\text{True} \vdash (\Rightarrow)^n \text{False})\]

where \((\Rightarrow)^n\) is short for \(\Rightarrow\) being nested \(n\) times.

The reason we want a stronger consistency than the usual \(\neg(\text{True} \vdash \text{False})\) is our modalities: it should be impossible to derive a contradiction below the modalities. For \(\Box\) and \(\blacksquare\), this follows from the elimination rules. For updates, we use the fact that \(\Rightarrow \text{False} \vdash \Rightarrow \blacksquare \text{False} \Rightarrow \text{False}\). However, there is no elimination rule for \(\Rightarrow\), so we declare that it is impossible to derive a contradiction below any number of laters.
6 Model and Semantics

The semantics closely follows the ideas laid out in [2].

Semantic domains. The semantic domains are interpreted as follows:

\[
\begin{align*}
\llbracket iProp \rrbracket & \triangleq UPred(M) \\
\llbracket \tau + \tau' \rrbracket & \triangleq \llbracket \tau \rrbracket + \llbracket \tau' \rrbracket \\
\llbracket \tau \times \tau' \rrbracket & \triangleq \llbracket \tau \rrbracket \times \llbracket \tau' \rrbracket \\
\llbracket \tau \rightarrow \tau' \rrbracket & \triangleq \llbracket \tau \rrbracket \rightarrow^* \llbracket \tau' \rrbracket \\
\llbracket 0 \rrbracket & \triangleq \Delta \emptyset \\
\llbracket 1 \rrbracket & \triangleq \Delta \{()\}
\end{align*}
\]

For the remaining base types \( \tau \) defined by the signature \( S \), we pick an object \( X_\tau \) in \( OFE \) and define

\[
\llbracket \tau \rrbracket \triangleq X_\tau
\]

For each function symbol \( F : \tau_1, \ldots, \tau_n \rightarrow \tau_{n+1} \in F \), we pick a function \( \llbracket F \rrbracket : \llbracket \tau_1 \rrbracket \times \cdots \times \llbracket \tau_n \rrbracket \rightarrow^* \llbracket \tau_{n+1} \rrbracket \).

Interpretation of propositions.

Remember that \( UPred(M) \) is isomorphic to \( M \overset{\text{mon}}{\rightarrow^*} SProp \). We are thus going to define the propositions as mapping camera elements to sets of step-indices.

\[
\begin{align*}
\llbracket \Gamma \vdash t =_\tau u : iProp \rrbracket_\rho & \triangleq \lambda n. \left\{ \begin{array}{ll}
\llbracket \Gamma \vdash t : \tau \rrbracket_\rho & \llbracket \Gamma \vdash u : \tau \rrbracket_\rho
\end{array} \right\} \\
\llbracket \Gamma \vdash \text{False} : iProp \rrbracket_\rho & \triangleq \lambda n. \emptyset \\
\llbracket \Gamma \vdash \text{True} : iProp \rrbracket_\rho & \triangleq \lambda n. \mathbb{N} \\
\llbracket \Gamma \vdash P \land Q : iProp \rrbracket_\rho & \triangleq \lambda a. \left( \llbracket \Gamma \vdash P : iProp \rrbracket_\rho(a) \cap \llbracket \Gamma \vdash Q : iProp \rrbracket_\rho(a) \right) \\
\llbracket \Gamma \vdash P \lor Q : iProp \rrbracket_\rho & \triangleq \lambda a. \left( \llbracket \Gamma \vdash P : iProp \rrbracket_\rho(a) \cup \llbracket \Gamma \vdash Q : iProp \rrbracket_\rho(a) \right) \\
\llbracket \Gamma \vdash P \Rightarrow Q : iProp \rrbracket_\rho & \triangleq \lambda a. \left\{ n \left| \begin{array}{l}
m \in \llbracket \Gamma \vdash P : iProp \rrbracket_\rho(b) \\
m \in \llbracket \Gamma \vdash Q : iProp \rrbracket_\rho(b)
\end{array} \right\} \right\} \\
\llbracket \Gamma \vdash \forall x : \tau. P : iProp \rrbracket_\rho & \triangleq \lambda a. \left\{ n \left| \forall v \in \llbracket \tau \rrbracket. n \in \llbracket \Gamma, x : \tau \vdash P : iProp \rrbracket_\rho[x \leftarrow v](a) \right\} \right\} \\
\llbracket \Gamma \vdash \exists x : \tau. P : iProp \rrbracket_\rho & \triangleq \lambda a. \left\{ n \left| \exists v \in \llbracket \tau \rrbracket. n \in \llbracket \Gamma, x : \tau \vdash P : iProp \rrbracket_\rho[x \leftarrow v](a) \right\} \right\}
\end{align*}
\]
Interpretation of non-propositional terms

\[
\llbracket \Gamma \vdash P \rightarrow Q \rbracket_{\rho} \triangleq \lambda a. \begin{cases}
\forall m, b, m \leq n \land m \in \mathcal{V}(a \cdot b) \Rightarrow \\
m \in \llbracket \Gamma \vdash P \rbracket_{\rho}(b) \Rightarrow \\
\end{cases}
\]

For every definition, we have to show all the side-conditions: The maps have to be non-expansive and monotone.

**Interpretation of non-propositional terms**

\[
\llbracket \Gamma \vdash t : \tau \rbracket_{\rho} : [\Gamma] \rightarrow [\tau]
\]

\[
\llbracket \Gamma \vdash x : \tau \rbracket_{\rho} \triangleq \rho(x)
\]

\[
\llbracket \Gamma \vdash F(t_1, \ldots, t_n) : \tau_{n+1} \rbracket_{\rho} \triangleq \llbracket F \rbracket([\llbracket \Gamma \vdash t_1 : \tau_1 \rbracket_{\rho}, \ldots, \llbracket \Gamma \vdash t_n : \tau_n \rbracket_{\rho}])
\]

\[
\llbracket \Gamma \vdash \lambda x : \tau. t : \tau \rbracket_{\rho} \triangleq \lambda u : [\tau]. \llbracket \Gamma, x : \tau \vdash t : \tau \rbracket_{\rho[x \leftarrow u]}
\]

\[
\llbracket \Gamma \vdash \text{fix}_{\mathcal{V}}(\lambda u : [\tau]. \llbracket \Gamma, x : \tau \vdash t : \tau \rbracket_{\rho[x \leftarrow u]}) \rbracket_{\rho} \triangleq \text{fix}_{\mathcal{V}}(\lambda u : [\tau]. \llbracket \Gamma, x : \tau \vdash t : \tau \rbracket_{\rho[x \leftarrow u]})
\]

\[
\llbracket \Gamma \vdash \text{abort} t : \tau \rbracket_{\rho} \triangleq \text{abort}_{[\tau]}([\llbracket \Gamma \vdash t : 0 \rbracket_{\rho}]
\]

\[
\llbracket \Gamma \vdash () : 1 \rbracket_{\rho} \triangleq ()
\]

\[
\llbracket \Gamma \vdash (t_1, t_2) : \tau_1 \times \tau_2 \rbracket_{\rho} \triangleq ([\llbracket \Gamma \vdash t_1 : \tau_1 \rbracket_{\rho}, [\llbracket \Gamma \vdash t_2 : \tau_2 \rbracket_{\rho}]
\]

\[
\llbracket \Gamma \vdash \pi_1 t : \tau_1 \rbracket_{\rho} \triangleq \pi_1([\llbracket \Gamma \vdash t : \tau_1 \times \tau_2 \rbracket_{\rho})
\]

\[
\llbracket \Gamma \vdash \text{inj}_1 t : \tau_1 + \tau_2 \rbracket_{\rho} \triangleq \text{inj}_1([\llbracket \Gamma \vdash t : \tau_1 \rbracket_{\rho})
\]

\[
\llbracket \Gamma \vdash \text{match } t \text{ with } \text{inj}_1 x_1, t_1 | \text{inj}_2 x_2, t_2 \text{ end} : \tau \rbracket_{\rho} \triangleq [\llbracket \Gamma, x_1 : \tau_1 \vdash t_1 : \tau \rbracket_{\rho[x_1 \leftarrow u}]
\]

where \( [\llbracket \Gamma \vdash t : \tau_1 + \tau_2 \rbracket_{\rho} = \text{inj}_1(u) \)

\[
[\llbracket \alpha : M \rbracket_{\rho} \triangleq \alpha
\]

\[
[\llbracket \Gamma \vdash t : M \rbracket_{\rho} \triangleq [\llbracket \Gamma \vdash t : M \rbracket_{\rho}]
\]

\[
[\llbracket \Gamma \vdash t : u : M \rbracket_{\rho} \triangleq [\llbracket \Gamma \vdash t : M \rbracket_{\rho} \cdot [\llbracket \Gamma \vdash u : M \rbracket_{\rho}]
\]

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An environment \( \Gamma \) is interpreted as the set of finite partial functions \( \rho \), with \( \text{dom}(\rho) = \text{dom}(\Gamma) \) and \( \rho(x) \in \llbracket \Gamma(x) \rrbracket \). Above, \( \text{fix} \) is Banach’s fixed-point (see Theorem 1), and \( \text{abort}_T \) is the unique function \( \emptyset \to T \).

**Logical entailment.** We can now define *semantic* logical entailment.

**Interpretation of entailment**

\[
\llbracket \Gamma \mid \Theta \vdash P \rrbracket : \text{Prop}
\]

\[
\llbracket \Gamma \mid P \vdash Q \rrbracket \triangleq \forall n \in \mathbb{N}. \forall r \in M. \forall \rho \in \llbracket \Gamma \rrbracket,
\]

\[
n \in \mathcal{V}(r) \land n \in \llbracket \Gamma \vdash P : \text{iProp} \rrbracket, \rho(r) \Rightarrow n \in \llbracket \Gamma \vdash Q : \text{iProp} \rrbracket, \rho(r)
\]

The following soundness theorem connects syntactic and semantic entailment. It is proven by showing that all the syntactic proof rules of §5 can be validated in the model.

\[
\Gamma \mid P \vdash Q \Rightarrow \llbracket \Gamma \mid P \vdash Q \rrbracket
\]

It now becomes straight-forward to show consistency of the logic.
7 Extensions of the Base Logic

In this section we discuss some additional constructions that we define within and on top of the base logic. These are not “extensions” in the sense that they change the proof power of the logic, they just form useful derived principles.

7.1 Derived Rules about Base Connectives

We collect here some important and frequently used derived proof rules.

\[ P \Rightarrow Q \vdash P \] \[ P * \exists x. Q \vdash \exists x. P * Q \] \[ P * \forall x. Q \vdash \forall x. P * Q \]

Verifying that existential quantifiers commute with separating conjunction requires an intermediate step using a magic wand: From \( P * \exists x. Q \vdash \exists x. P * Q \) we can deduce \( \exists x. Q \vdash P * \exists x. P * Q \) and then proceed via \( \exists \)-elimination.

7.2 Derived Rules about Modalities

Iris comes with 4 built-in modalities (\( \square \), \( \blacksquare \), \( \Rightarrow \) and \( \triangleright \)) and, as we will see, plenty of derived modalities. However, almost all of them fall into one of two categories (except for \( \triangleright \), as we will see): they are either always-style modalities (“something holds in all/many (future) worlds”) or eventually-style modalities (“something holds in a possible (future) world”).

Eventually-style modalities are characterized by being easy to “add”/introduce, but hard to “remove”/eliminate. Consider, for example, the basic update modality \( \Rightarrow \): we have \( P \vdash \Rightarrow P \) (\( \text{upd-I} \)), but the inverse direction does not hold. Instead, from \( \text{udp-mono} \) and \( \text{udp-trans} \), we can derive the following elimination principle:

\[ \begin{align*}
P & \vdash \Rightarrow Q \\
\Rightarrow P & \vdash \Rightarrow Q
\end{align*} \]

In other words, we can remove an \( \Rightarrow \) in front of an assumption if the goal is itself wrapped in \( \Rightarrow \). Another way to view this rule is to think of it as a bind rule. Indeed, together with \( \text{udp-I} \), this rule shows that \( \Rightarrow \) forms a monad.

Always-style modalities, on the other hand, are easy to “remove”/eliminate, but hard to “add”/introduce. The most widely used example of that in Iris is the persistence modality \( \square \): we have \( \square P \vdash P \) (\( \square \)-E), but the inverse direction does not hold. Instead, from \( \square \)-mono and \( \square P \vdash \square \square P \), we can derive the following introduction principle:

\[ \begin{align*}
\square P & \vdash Q \\
\square P & \vdash \square Q
\end{align*} \]

In other words, we can remove an \( \square \) from the goal if all our assumptions are wrapped in \( \square \). This matches the algebraic structure of a comonad.

In particular, both eventually-style and always-style modalities are idempotent: we have \( \Rightarrow \Rightarrow P \vdash \Rightarrow P \) and \( \square \square P \vdash \square P \)
Beyond this, all modalities come with plenty of rules that show how they commute around other connectives and modalities. And, of course, they come with a few “defining rules” that give the modalities their individual meaning, i.e., for the update modality, that would be \texttt{UPD-UPDATE}.

In the following, we briefly discuss each of the modalities.

**Update modality.** As already mentioned, the update modality is an eventually-style modality:

\[
\text{update-E} \quad \frac{P \vdash \mathcal{R} Q}{\mathcal{R} P \vdash \mathcal{R} Q} \quad \text{update-idemp} \quad \mathcal{R} P \vdash \mathcal{R} P
\]

Beyond this (and the obvious variant of \texttt{update-frame} that exploits commutativity of separating conjunction), there are no outstandingly interesting derived rules.

**Persistence modality.** As already mentioned, the persistence modality is an always-style modality:

\[
\text{always-I} \quad \frac{P \vdash Q}{\mathcal{A} P \vdash \mathcal{A} Q} \quad \text{always-idemp} \quad \mathcal{A} P \vdash \mathcal{A} P
\]

Some further interesting derived rules include:

\[
\mathcal{A}(P \land Q) \vdash \mathcal{A} P \land \mathcal{A} Q \quad \mathcal{A}(P \lor Q) \vdash \mathcal{A} P \lor \mathcal{A} Q \quad \mathcal{A} \text{True} \vdash \text{True} \quad \mathcal{A} \text{False} \vdash \text{False}
\]

\[
\mathcal{A}(P * Q) \vdash \mathcal{A} P * \mathcal{A} Q \quad \mathcal{A} P * Q \vdash \mathcal{A} P \land \mathcal{A} Q \quad \mathcal{A}(P \rightarrow Q) \vdash \mathcal{A} (P \Rightarrow Q)
\]

\[
\mathcal{A}(P \Rightarrow Q) \vdash \mathcal{A} P \Rightarrow \mathcal{A} Q \quad \mathcal{A}(P \rightarrow Q) \vdash \mathcal{A} P \rightarrow \mathcal{A} Q
\]

In particular, the persistence modality commutes around conjunction, disjunction, separating conjunction as well as universal and existential quantification. Commuting around conjunction can be derived from the primitive rule that says it commutes around universal quantification (as conjunction is equivalent to a universal quantification of a Boolean), and similar for disjunction. \texttt{True} \vdash \mathcal{A} \text{True} (which is basically persistence “commuting around” the nullary operator \texttt{True}) can be derived via \mathcal{A} commuting with universal quantification ranging over the empty type. A similar rule holds for \texttt{False}.

Moreover, if (at least) one conjunct is below the persistence modality, then conjunction and separating conjunction coincide.

**Plainness modality.** The plainness modality is very similar to the persistence modality (in fact, we have \(\blacksquare P \vdash \square P\), but the inverse does not hold). It is always-style:

\[
\blacksquare \text{-I} \quad \frac{\blacksquare P \vdash Q}{\square P \vdash \blacksquare Q} \quad \blacksquare \blacksquare P \vdash \blacksquare P
\]

It also commutes around separating conjunction, conjunction, disjunction, universal and existential quantification (and \texttt{True} and \texttt{False}).
The key difference to the persistence modality □ is that ■ provides a \textit{propositional extensionality} principle:

\[
\blacksquare ((P \Rightarrow Q) \land (Q \Rightarrow P)) \vdash P =_{\text{Prop}} Q
\]

In contrast, □ permits using some forms of ghost state (\textit{Own} \((a) \vdash \square \text{Own} \((a))\)).

Having both of these principles for the same modality would lead to a contradiction: imagine we have an RA with elements \(a, b\) such that \(|a|\) is incompatible with \(b\) (i.e., \(\neg \forall ([a] \cdot b)\)). Then we can prove:

\[
\text{Own} \((|a|) \vdash \square \text{Own} \((|a|) \land \text{Own} (b) \vdash \text{Own} \((|a| \cdot \text{Own} (b) \vdash \forall ([a] \cdot b).\)
\]

But now, if we had propositional extensionality for □ the way we do for ■, we could deduce \(\text{False} =_{\text{Prop}} \text{Own} (b)\), and that is clearly wrong. This issue arises because □, as we have seen, still lets us use some resources from the context, while propositional equality has to hold completely disregarding current resources.

\textbf{Later modality.} The later modality is the “odd one out” in the sense that it is neither eventually-style nor always-style, because it is not idempotent.\(^3\) with \(\Rightarrow\), the number of times the modality is applied matters, and we can get rid of \textit{exactly one} layer of \(\Rightarrow\) in the assumptions only by doing the same in the conclusion (\(\Rightarrow\)-\textit{MONO}).

Some derived rules:

\[
\blacksquare (P \Rightarrow Q) \vdash P \Rightarrow Q \quad \blacksquare (P \Rightarrow Q) \vdash P \Rightarrow \blacksquare Q \quad \blacksquare (P \Rightarrow Q) \vdash \blacksquare P \Rightarrow Q
\]

\(\blacksquare (P \land Q) \vdash P \land Q\) \(\blacksquare (P \lor Q) \vdash P \lor Q\)

\(\tau\) is inhabited\(^4\)

\[
\blacksquare (\exists x : \tau. P) \vdash \exists x : \tau. \blacksquare P
\]

\(\blacksquare \text{True} \vdash \text{True}\)

\(\blacksquare (P \Rightarrow Q) \vdash P \Rightarrow \blacksquare Q\) \(\blacksquare (P \Rightarrow Q) \vdash \blacksquare P \Rightarrow Q\)

\textbf{7.3 Persistent Propositions}

We call a proposition \(P\) \textit{persistent} if \(P \vdash \square P\). These are propositions that “do not own anything”, so we can (and will) treat them like “normal” intuitionistic propositions.

Of course, \(\square P\) is persistent for any \(P\). Furthermore, by the proof rules given in §5.3, \textit{True}, \textit{False}, \(t = t'\) as well as \(\left[|a|\right]_1\) and \(\forall (a)\) are persistent. Persistence is preserved by conjunction, disjunction, separating conjunction as well as universal and existential quantification and \(\Rightarrow\).

\textsuperscript{3}This means \(\Rightarrow\) is neither a monad nor a comonad—it does form an applicative functor, though.

\textsuperscript{4}Also see https://en.wikipedia.org/wiki/L%C3%B6b%27s_theorem.
7.4 Timeless Propositions and Except-0

One of the troubles of working in a step-indexed logic is the “later” modality \( \triangleright \). It turns out that we can somewhat mitigate this trouble by working below the following except-0 modality:

\[
\Box P \triangleq \neg \top \lor P
\]

Except-0 satisfies the usual laws of a “monadic” modality (similar to, e.g., the update modalities):

\[
\begin{align*}
\text{ex0-mono} & : P \vdash Q \quad \text{ex0-intro} : P \vdash \Box P \\
\text{ex0-idem} & : P \vdash \Box P \\
\text{ex0-timeless-strip} & : \text{timeless}(P) \vdash \Box P
\end{align*}
\]

In particular, from \text{ex0-mono} and \text{ex0-idem} we can derive a “bind”-like elimination rule:

\[
\begin{align*}
\text{ex0-elim} & : P \vdash Q \\
\Box P \vdash \Box Q
\end{align*}
\]

This modality is useful because there is a class of propositions which we call timeles propositions, for which we have

\[
\text{timeless}(P) \triangleq \triangleright P \vdash P
\]

In other words, when working below the except-0 modality, we can strip away the later from timeles propositions (using \text{ex0-elim}):

\[
\begin{align*}
\text{ex0-timeless-strip} & : \text{timeless}(P) \vdash \Box P \\
\triangleright P \vdash \Box Q
\end{align*}
\]

In fact, it turns out that we can strip away later from timeles propositions even when working under the later modality:

\[
\begin{align*}
\text{later-timeless-strip} & : \text{timeless}(P) \vdash \Box P \\
\triangleright P \vdash \Box Q
\end{align*}
\]

This follows from \( \triangleright P \vdash \Box \top \lor P \), and then by straightforward disjunction elimination.

The following rules identify the class of timeles propositions:

\[
\begin{align*}
\Gamma \vdash \text{timeless}(P) & \quad \Gamma \vdash \text{timeless}(Q) \quad \Gamma \vdash \text{timeless}(P \land Q) \\
\Gamma \vdash \text{timeless}(P) & \quad \Gamma \vdash \text{timeless}(Q) \quad \Gamma \vdash \text{timeless}(P \lor Q) \\
\Gamma \vdash \text{timeless}(P) & \quad \Gamma \vdash \text{timeless}(Q) \quad \Gamma \vdash \text{timeless}(P \rightarrow Q) \\
\Gamma \vdash \text{timeless}(P) & \quad \Gamma \vdash \text{timeless}(Q) \quad \Gamma \vdash \text{timeless}(P \leftrightarrow Q)
\end{align*}
\]
7.5 Dynamic Composeable Higher-Order Resources

The base logic described in §5 works over an arbitrary camera $M$ defining the structure of the resources. It turns out that we can generalize this further and permit picking cameras “$\Sigma(i\text{Prop})$” that depend on the structure of propositions themselves. Of course, $i\text{Prop}$ is just the syntactic type of propositions; for this to make sense we have to look at the semantics.

Furthermore, there is a composability problem with the given logic: if we have one proof performed with camera $M_1$, and another proof carried out with a different camera $M_2$, then the two proofs are actually carried out in two entirely separate logics and hence cannot be combined.

Finally, in many cases just having a single “instance” of a camera available for reasoning is not enough. For example, when reasoning about a dynamically allocated data structure, every time a new instance of that data structure is created, we will want a fresh resource governing the state of this particular instance. While it would be possible to handle this problem whenever it comes up, it turns out to be useful to provide a general solution.

The purpose of this section is to describe how we solve these issues.

Picking the resources. The key ingredient that we will employ on top of the base logic is to give some more fixed structure to the resources. To instantiate the logic with dynamic higher-order ghost state, the user picks a family of locally contractive bifunctors $(\Sigma_i: \text{COFE}^{\text{op}} \times \text{COFE} \to \text{Camera})_{i \in I}$. (This is in contrast to the base logic, where the user picks a single, fixed camera that has a unit.)

From this, we construct the bifunctor defining the overall resources as follows:

$$GName \triangleq \mathbb{N}$$

$$\text{ResF}(T^{\text{op}}, T) \triangleq \prod_{i \in I} GName \overset{\text{fin}}{\longrightarrow} \Sigma_i(T^{\text{op}}, T)$$

We will motivate both the use of a product and the finite partial function below. $\text{ResF}(T^{\text{op}}, T)$ is a camera by lifting the individual cameras pointwise, and it has a unit (using the empty finite partial function). Furthermore, since the $\Sigma_i$ are locally contractive, so is $\text{ResF}$.

Now we can write down the recursive domain equation:

$$i\text{PreProp} \cong \text{UPred}(\text{ResF}(i\text{PreProp}, i\text{PreProp}))$$

Here, $i\text{PreProp}$ is a COFE defined as the fixed-point of a locally contractive bifunctor, which exists.
and is unique up to isomorphism by Theorem 2, so we obtain some object \( i\text{PreProp} \) such that:

\[
\text{Res} \triangleq \text{Res}_F(i\text{PreProp}, i\text{PreProp}) \\
i\text{Prop} \triangleq \text{UPred}(\text{Res}) \\
\xi : i\text{Prop} \xrightarrow{\text{ne}} i\text{PreProp} \\
\xi^{-1} : i\text{PreProp} \xrightarrow{\text{ne}} i\text{Prop} \\
\xi(\xi^{-1}(x)) \triangleq x \\
\xi^{-1}(\xi(x)) \triangleq x
\]

Now we can instantiate the base logic described in §5 with \( \text{Res} \) as the chosen camera:

\[
\llbracket i\text{Prop} \rrbracket \triangleq \text{UPred}(\text{Res})
\]

We obtain that \( \llbracket i\text{Prop} \rrbracket = i\text{Prop} \). Effectively, we just defined a way to instantiate the base logic with \( \text{Res} \) as the camera of resources, while providing a way for \( \text{Res} \) to depend on \( i\text{PreProp} \), which is isomorphic to \( \llbracket i\text{Prop} \rrbracket \).

We thus obtain all the rules of §5, and furthermore, we can use the maps \( \xi \) and \( \xi^{-1} \) in the logic to convert between logical propositions \( \llbracket i\text{Prop} \rrbracket \) and the domain \( i\text{PreProp} \) which is used in the construction of \( \text{Res} \) — so from elements of \( i\text{PreProp} \), we can construct elements of \( \llbracket M \rrbracket \), which are the elements that can be owned in our logic.

**Proof composability.** To make our proofs composeable, we generalize our proofs over the family of functors. This is possible because we made \( \text{Res} \) a product of all the cameras picked by the user, and because we can actually work with that product “pointwise”. So instead of picking a concrete family, proofs will assume to be given an arbitrary family of functors, plus a proof that this family contains the functors they need. Composing two proofs is then merely a matter of conjoining the assumptions they make about the functors. Since the logic is entirely parametric in the choice of functors, there is no trouble reasoning without full knowledge of the family of functors.

Only when the top-level proof is completed we will “close” the proof by picking a concrete family that contains exactly those functors the proof needs.

**Dynamic resources.** Finally, the use of finite partial functions lets us have as many instances of any camera as we could wish for: Because there can only ever be finitely many instances already allocated, it is always possible to create a fresh instance with any desired (valid) starting state. This is best demonstrated by giving some proof rules.

So let us first define the notion of ghost ownership that we use in this logic. Assuming that the family of functors contains the functor \( \Sigma_i \) at index \( i \), and furthermore assuming that \( M_i = \Sigma_i(i\text{PreProp}, i\text{PreProp}) \), given some \( a \in M_i \) we define:

\[
[\{a : M_i\}]^\top \triangleq \text{Own}((\ldots, 0, i : [\gamma \leftarrow a], 0, \ldots))
\]

This is ownership of the pair (element of the product over all the functors) that has the empty finite partial function in all components except for the component corresponding to index \( i \), where we own the element \( a \) at index \( \gamma \) in the finite partial function.

We can show the following properties for this form of ownership:
Below, we will always work within (an instance of) the logic as described here. Whenever a camera is used in a proof, we implicitly assume it to be available in the global family of functors. We will typically leave the $M_i$ implicit when asserting ghost ownership, as the type of $a$ will be clear from the context.
8 Language

A language $\Lambda$ consists of a set $\text{Expr}$ of expressions (metavariable $e$), a set $\text{Val}$ of values (metavariable $v$), a set $\text{Obs}$ of observations\(^5\) (or “observable events”) and a set $\text{State}$ of states (metavariable $\sigma$) such that

- There exist functions $\text{val_to_expr} : \text{Val} \rightarrow \text{Expr}$ and $\text{expr_to_val} : \text{Expr} \rightarrow \text{Val}$ (notice the latter is partial), such that
  \[ \forall e, v. \text{expr_to_val}(e) = v \Rightarrow \text{val_to_expr}(v) = e \]
  \[ \forall v. \text{expr_to_val}(\text{val_to_expr}(v)) = v \]

- There exists a primitive reduction relation
  \[ (\_ , \_ \rightarrow^t \_ , \_ , \_ ) \subseteq (\text{Expr} \times \text{State}) \times \text{List}(\text{Obs}) \times (\text{Expr} \times \text{State} \times \text{List}(\text{Expr})) \]

A reduction $e_1, \sigma_1 \rightarrow^t e_2, \sigma_2, \vec{e}$ indicates that, when $e_1$ in state $\sigma_1$ reduces to $e_2$ with new state $\sigma_2$, the new threads in the list $\vec{e}$ is forked off and the observations $\vec{\kappa}$ are made. We will write $e_1, \sigma_1 \rightarrow_t e_2, \sigma_2$ for $e_1, \sigma_1 \rightarrow^t e_2, \sigma_2, ()$, i.e., when no threads are forked off and no observations are made.

- All values are stuck:
  \[ e, \_ \rightarrow^t \_ , \_ , \_ \Rightarrow \text{expr_to_val}(e) = \bot \]

**Definition 19.** An expression $e$ and state $\sigma$ are reducible (written $\text{red}(e, \sigma)$) if

\[ \exists \vec{\kappa}, e_2, \sigma_2, \vec{e}, e, \sigma \rightarrow^t \vec{e}, \sigma_2, \vec{e} \]

**Definition 20.** An expression $e$ is weakly atomic if it reduces in one step to something irreducible:

\[ \text{atomic}(e) \triangleq \forall \sigma_1, \vec{\kappa}, e_2, \sigma_2, \vec{e}, e, \sigma \rightarrow^t \vec{e}, \sigma_2, \vec{e} \Rightarrow \neg \text{red}(e_2, \sigma_2) \]

It is strongly atomic if it reduces in one step to a value:

\[ \text{strongly_atomic}(e) \triangleq \forall \sigma_1, \vec{\kappa}, e_2, \sigma_2, \vec{e}, e, \sigma \rightarrow^t \vec{e}, \sigma_2, \vec{e} \Rightarrow \text{expr_to_val}(e_2) \neq \bot \]

We need two notions of atomicity to accommodate both kinds of weakest preconditions that we will define later: If the weakest precondition ensures that the program cannot get stuck, weak atomicity is sufficient. Otherwise, we need strong atomicity.

**Definition 21 (Context).** A function $K : \text{Expr} \rightarrow \text{Expr}$ is a context if the following conditions are satisfied:

1. $K$ does not turn non-values into values:

\[ \forall e. \text{expr_to_val}(e) = \bot \Rightarrow \text{expr_to_val}(K(e)) = \bot \]

\[^5\text{See } \text{https://gitlab.mpi-sws.org/iris/iris/merge_requests/173 } \text{for how observations are useful to encode prophecy variables.}\]
2. One can perform reductions below \( K \):
\[
\forall e_1, \sigma_1, \vec{\kappa}, e_2, \sigma_2, \vec{\epsilon}. e_1 \xrightarrow{\vec{\kappa}} e_2, \sigma_2, \vec{\epsilon} \Rightarrow K(e_1), \sigma_1 \xrightarrow{\vec{\kappa}} K(e_2), \sigma_2, \vec{\epsilon}
\]

3. Reductions stay below \( K \) until there is a value in the hole:
\[
\forall e'_1, \sigma_1, \vec{\kappa}, e_2, \sigma_2, \vec{\epsilon}. \text{expr_to_val}(e'_1) = \bot \land K(e'_1), \sigma_1 \xrightarrow{\vec{\kappa}} e_2, \sigma_2, \vec{\epsilon} \Rightarrow \\
\exists e_2 = K(e'_2) \land e'_1, \sigma_1 \xrightarrow{\vec{\kappa}} e'_2, \sigma_2, \vec{\epsilon}
\]

### 8.1 Concurrent Language

For any language \( \Lambda \), we define the corresponding thread-pool semantics.

**Machine syntax**

\[
T \in \text{ThreadPool} \triangleq \text{List(Expr)}
\]

**Machine reduction**

\[
\frac{e_1, \sigma_1 \xrightarrow{\vec{\kappa}} e_2, \sigma_2, \vec{\epsilon}}{T \xrightarrow{\text{tp}} T' \; \; ; \; \; \sigma'}
\]

We use \( \xrightarrow{\text{tp}}^* \) for the reflexive transitive closure of \( \xrightarrow{\text{tp}} \), as usual concatenating the lists of observations of the individual steps.
9 Program Logic

This section describes how to build a program logic for an arbitrary language (c.f. §8) on top of the base logic. So in the following, we assume that some language $\Lambda$ was fixed. Furthermore, we work in the logic with higher-order ghost state as described in §7.5.

9.1 World Satisfaction, Invariants, Fancy Updates

To introduce invariants into our logic, we will define weakest precondition to explicitly thread through the proof that all the invariants are maintained throughout program execution. However, in order to be able to access invariants, we will also have to provide a way to temporarily disable (or “open”) them. To this end, we use tokens that manage which invariants are currently enabled.

We assume to have the following four cameras available:

\[
\begin{align*}
\text{InvName} & \triangleq \mathbb{N} \\
\text{INV} & \triangleq \text{AUTH}(\text{InvName} \xrightarrow{\text{fin}} \text{AG}(\text{PreProp})) \\
\text{EN} & \triangleq \phi(\text{InvName}) \\
\text{DIS} & \triangleq \phi^{\text{fin}}(\text{InvName})
\end{align*}
\]

The last two are the tokens used for managing invariants, INV is the monoid used to manage the invariants themselves.

We assume that at the beginning of the verification, instances named $\gamma_{\text{State}}, \gamma_{\text{INV}}, \gamma_{\text{EN}}$ and $\gamma_{\text{DIS}}$ of these cameras have been created, such that these names are globally known.

**World Satisfaction.** We can now define the proposition $W$ (world satisfaction) which ensures that the enabled invariants are actually maintained:

\[
W \triangleq \exists I : \text{InvName} \xrightarrow{\text{fin}} \text{iProp} \cdot \left\{ \begin{array}{l}
\forall \iota \in \text{dom}(I) \cdot \langle I(\iota) \rangle \\
\forall \iota \in \text{dom}(I) \cdot \langle I(\iota) \rangle \\
\end{array} \right\}^{\text{INV}}
\]

**Invariants.** The following proposition states that an invariant with name $\iota$ exists and maintains proposition $P$:

\[
[P] \triangleq \left\{ \begin{array}{l}
\forall \iota \in \text{InvName} \cdot \text{ag}(\text{PreProp}) \\
\forall \iota \in \text{InvName} \cdot \text{ag}(\text{PreProp}) \\
\end{array} \right\}^{\text{INV}}
\]

**Fancy Updates and View Shifts.** Next, we define fancy updates, which are essentially the same as the basic updates of the base logic (§5), except that they also have access to world satisfaction and can enable and disable invariants:

\[
E_1 \models E_2 P \triangleq W \ast \left( E_2 \ast E_1 \right)^{\text{EN}} \
\]

Here, $E_1$ and $E_2$ are the masks of the view update, defining which invariants have to be (at least!) available before and after the update. We use $\top$ as symbol for the largest possible mask, $\mathbb{N}$, and $\bot$ for the smallest possible mask $\emptyset$. We will write $\models E P$ for $E_1 \models E_2 P$. Fancy updates satisfy the following basic proof rules:
We can further define the notions of view shifts and linear view shifts:

\[ P \equiv_{\exists} Q \triangleq P \equiv_{\exists} Q \]

These two are useful when writing down specifications and for comparing with previous versions of Iris, but for reasoning, it is typically easier to just work directly with fancy updates. Still, just to give an idea of what view shifts “are”, here are some proof rules for them:

9.2 Weakest Precondition

Finally, we can define the core piece of the program logic, the proposition that reasons about program behavior: Weakest precondition, from which Hoare triples will be derived.

Defining weakest precondition. We assume that everything making up the definition of the language, i.e., values, expressions, states, the conversion functions, reduction relation and all their properties, are suitably reflected into the logic (i.e., they are part of the signature $\mathcal{S}$). We further assume (as a parameter) a predicate $S : \text{State} \times \text{List} (\text{Obs}) \times \mathbb{N} \rightarrow \text{iProp}$ that interprets the machine state as an Iris proposition, and a predicate $\Phi_F : \text{Val} \rightarrow \text{iProp}$ that serves as postcondition for forked-off threads. The state interpretation can depend on the current physical state, the list of future observations as well as the total number of forked threads (that is one less that the total number of threads). This can be instantiated, for example, with ownership of an authoritative RA to tie the physical state to fragments that are used for user-level proofs. Finally, weakest precondition
takes a parameter \( s \in \{\text{NotStuck}, \text{Stuck}\} \) indicating whether program execution is allowed to get stuck.

\[
wp(S, \Phi_F, s) \triangleq \mu \text{wp}_{\text{rec}}. \lambda E, e, \Phi.
(\exists v. \text{expr\_to\_val}(e) = v \land \models_E \Phi(v)) \lor
\]
\[
(\text{expr\_to\_val}(e) = \bot \land \forall \sigma, \vec{r}, \vec{r}', n. S(\sigma, \vec{r} + \vec{r}', n) \models_E \vec{r}^0)
\]
\[
(s = \text{NotStuck} \Rightarrow \text{red}(e, \sigma)) \star \forall \vec{e}', \vec{e}. (e, \sigma \rightarrow_{\vec{e}} e', \sigma', \vec{e}) \models_E \vec{r}^0 \Rightarrow \models_E
\]
\[
S(\sigma', \vec{r}', n + |\vec{e}|) \star \text{wp}_{\text{rec}}(E, e', \Phi) \star \bigstar_{e \in E} \text{wp}_{\text{rec}}(\top, e'', \Phi_F)
\]
\[
\text{wp}^{s; \Phi_F}_x \{v. P\} \triangleq \wp(S, \Phi_F, s)(E, e, \lambda v. P)
\]

The \( S \) and \( \Phi_F \) will always be set by the context; typically, when instantiating Iris with a language, we also pick the corresponding state interpretation \( S \) and fork-postcondition \( \Phi_F \). All proof rules leave \( S \) and \( \Phi_F \) unchanged. If we leave away the mask \( E \), we assume it to default to \( \top \). If we leave away the stickiness \( s \), it defaults to \( \text{NotStuck} \).

**Laws of weakest precondition.** The following rules can all be derived:

\[
\begin{array}{ccc}
\text{WP-VALUE} & P[v/x] \vdash \text{wp}_{s;E} v \{x. P\} & \text{WP-MONO} \\
\hline
\text{WP-FUP} & x : \text{val} \mid P \vdash Q & (s_2 = \text{Stuck} \lor s_1 = s_2) \\
\hline
\text{FUP-WP} & \models_E \text{wp}_{s;E} e \{x. P\} \vdash \text{wp}_{s;E} e \{x. P\} & \Gamma \mid \text{wp}_{s_1;E_1} e \{x. P\} \vdash \text{wp}_{s_2;E_2} e \{x. Q\}
\end{array}
\]

\[
\begin{array}{ccc}
\text{WP-ATOMIC} & s = \text{NotStuck} \Rightarrow \text{atomic}(e) & s = \text{Stuck} \Rightarrow \text{strongly\_atomic}(e) \\
\hline
\text{WP-FRAME} & \vdash \text{wp}_{s;E} e \{x. P\} \vdash \text{wp}_{s;E} e \{x. Q * P\} & \text{EXPR\_TO\_VAL}(e) = \bot \ \ \ \ \ \ \ \ \ \ E_2 \subseteq E_1
\end{array}
\]

\[
\begin{array}{ccc}
\text{WP-FRAME-STEP} & \text{expr\_to\_val}(e) = \bot & \text{expr\_to\_val}(e) = \bot \ \ \ \ \ \ \ \ \ \ E_2 \subseteq E_1
\end{array}
\]

\[
\begin{array}{ccc}
\text{WP-BIND} & \text{val\_to\_expr}(x) \{y. P\} & \text{K is a context}
\end{array}
\]

\[
\Gamma \vdash \text{wp}_{s;E} e \{x. \text{wp}_{s;E} K(\text{val\_to\_expr}(x)) \{y. P\}\} \vdash \text{wp}_{s;E} K(e) \{y. P\}
\]

We will also want a rule that connect weakest preconditions to the operational semantics of the language. This basically just copies the second branch (the non-value case) of the definition of weakest preconditions.
Adequacy of weakest precondition. The purpose of the adequacy statement is to show that our notion of weakest preconditions is realistic in the sense that it actually has anything to do with the actual behavior of the program. The most general form of the adequacy statement is about proving properties of an arbitrary program execution.

**Theorem 3** (Adequacy). Assume we are given some $e_1$, $\sigma_1$, $\vec{\kappa}$, $T_2$, $\sigma_2$ such that $(\llbracket e_1 \rrbracket, \sigma_1) \xrightarrow{*} (T_2, \sigma_2)$, and we are also given a meta-level property $p$ that we want to show. To verify that $p$ holds, it is sufficient to show the following Iris entailment:

$$\text{True} \vdash \mathbin{\Rightarrow}_{\top} \exists s, S, \Phi, \Phi_{F}. S(\sigma_1, \vec{\kappa}, 0) \ast wp_{S;\Phi_{F}}^{S;\Phi_{F}} e_1 \{x. \Phi(x)\} \ast \left( C_{S;\Phi_{F}}^{S;\Phi_{F}}(T_2, \sigma_2) \xrightarrow{*} \top \hat{p} \right)$$

where $C$ describes states that are consistent with the state interpretation and postconditions:

$$C_{S;\Phi_{F}}^{S;\Phi_{F}}(T_2, \sigma_2) \triangleq \exists e_2, T'_2, T_2 = \llbracket e_2 \rrbracket \ast$$

$$(s = \text{NotStuck} \Rightarrow \forall e \in T_2. \text{expr\_to\_val}(e) \neq \bot \lor \text{red}(e, \sigma_2)) \ast$$

$$S(\sigma_2, (), T'_2)\ast$$

$$(\text{expr\_to\_val}(e_2) \neq \bot \Rightarrow \Phi(\text{expr\_to\_val}(e_2)))\ast$$

$$\left( \bigast_{e \in T_2} \text{expr\_to\_val}(e) \neq \bot \Rightarrow \Phi_{F}(\text{expr\_to\_val}(e)) \right)$$

The $\hat{p}$ here arises because we need a way to talk about $p$ inside Iris. To this end, we assume that the signature $S$ contains some assertion $\hat{p}$:

$$\hat{p} : iProp \in F$$

Furthermore, we assume that the interpretation $\llbracket \hat{p} \rrbracket$ of $\hat{p}$ reflects $p$ (also see §6):

$$\llbracket \hat{p} \rrbracket : \llbracket iProp \rrbracket$$

$$\llbracket \hat{p} \rrbracket \triangleq \lambda_n. \{ n | p \}$$

The signature can of course state arbitrary additional properties of $\hat{p}$, as long as they are proven sound.

In other words, to show that $p$ holds, we have to prove an entailment in Iris that, starting from the empty context, chooses some state interpretation, postcondition, forked-thread postcondition and stuckness and then proves:

- the initial state interpretation,
• a weakest-precondition,
• and a view shift showing the desired $\hat{p}$ under the extra assumption $C(T_2, \sigma_2)$.

Notice that the state interpretation and the postconditions are chosen after doing a fancy update, which allows them to depend on the names of ghost variables that are picked in that initial fancy update. This gives us a chance to allocate some “global” ghost state that state interpretation and postcondition can refer to.

$C_s^{S;\Phi;\Phi_F}(T_2, \sigma_2)$ says that:

• The final thread-pool $T_2$ contains the final state of the main thread $e_2$, and any number of additional forked threads in $T_2'$.
• If this is a stuck-free weakest precondition, then all threads in the final thread-pool are either values or are reducible in the final state $\sigma_2$.
• The state interpretation $S$ holds for the final state.
• If the main thread reduced to a value, the post-condition $\Phi$ of the weakest precondition holds for that value.
• If any other thread reduced to a value, the forked-thread post-condition $\Phi_F$ holds for that value.

As an example for how to use this adequacy theorem, let us say we wanted to prove that a program $e_1$ for which we derived a $\text{NotStuck}$ weakest-precondition cannot get stuck:

**Corollary 1** (Stuck-freedom). Assume we are given some $e_1$ such that the following holds:

$$\text{True} \vdash \forall \sigma_1, \bar{\kappa},. \exists S, \Phi, S(\sigma_1, \bar{\kappa}, 0) \ast wp^{\text{NotStuck};\top} e_1 \{ x. \Phi(x) \}$$

Then it is the case that:

$$\forall \sigma_1, \bar{\kappa}, T_2, \sigma_2. ((e_1, \sigma_1) \xrightarrow{\text{tp}} (T_2, \sigma_2)) \Rightarrow \forall e \in T_2. \text{expr_to_val}(e) \neq \bot \lor \text{red}(e, \sigma_2)$$

To prove the conclusion of this corollary, we assume some $\sigma_1, \bar{\kappa}, T_2, \sigma_2$ and $(e_1, \sigma_1) \xrightarrow{\text{tp}} (T_2, \sigma_2)$, and we instantiate the main theorem with this execution and $p \equiv \forall e \in T_2. \text{expr_to_val}(e) \neq \bot \lor \text{red}(e, \sigma_2)$. We can then show the premise of adequacy using the Iris entailment that we assumed in the corollary and:

$$\text{True} \vdash C_{\text{NotStuck}}^{S;\Phi;\Phi_F}(T_2, \sigma_2) \xrightarrow{\text{Top}} p$$

This proof, just like the following, also exploits that we can freely swap between meta-level universal quantification ($\forall x. \text{True} \vdash P$) and quantification in Iris ($\text{True} \vdash \forall x. P$).

Similarly we could show that the postcondition makes adequate statements about the possible final value of the main thread:
Then it is the case that:

$$\forall \sigma, \bar{\kappa}, \exists s, S, \Phi_F \cdot S(\sigma, \bar{\kappa}, 0) \ast wp^{S,\Phi_F}_{s,t} e_1 \{x. x \in V\}$$

As a final example, we could use adequacy to show that the state interpretation is always in some set $\Sigma \subseteq State$:

**Corollary 3 (Adequate state interpretation).** Assume we are given some $e_1$ and a set $\Sigma \subseteq State$ such that the following holds (assuming we can talk about sets like $V$ inside the logic):

$$\forall \sigma, \bar{\kappa}, \exists s, S, \Phi_F \cdot S(\sigma, \bar{\kappa}, 0) \ast wp^{S,\Phi_F}_{s,t} e_1 \{\forall \sigma_2, m. S(\sigma_2, (,), m) \Rightarrow \sigma_2 \in \Sigma\}$$

Then it is the case that:

$$\forall \sigma, \bar{\kappa}, T_2, \sigma_2 \cdot ([e_1], \sigma_1) \xrightarrow{\kappa \ast}_{tp} ([val_to_expr(v_2)] + T_2, \sigma_2) \Rightarrow \sigma_2 \in \Sigma$$

To show this, we assume some $\sigma_1, \bar{\kappa}, v_2, T_2, \sigma_2$ such that $([e_1], \sigma_1) \xrightarrow{\kappa \ast}_{tp} ([val_to_expr(v_2)] + T_2, \sigma_2)$, and we instantiate adequacy with this execution and $p \triangleq v_2 \in Val$. Then we only have to show:

$$\forall \sigma, \bar{\kappa}, T_2, \sigma_2 \cdot ([e_1], \sigma_1) \xrightarrow{\kappa \ast}_{tp} ([val_to_expr(v_2)] + T_2, \sigma_2) \Rightarrow \sigma_2 \in \Sigma$$

**Hoare triples.** It turns out that weakest precondition is actually quite convenient to work with, in particular when performing these proofs in Coq. Still, for a more traditional presentation, we can easily derive the notion of a Hoare triple:

$$\{ P \} e \{ v. Q \}$$

We assume the state interpretation $S$ to be fixed by the context.

We only give some of the proof rules for Hoare triples here, since we usually do all our reasoning directly with weakest preconditions and use Hoare triples only to write specifications.

<table>
<thead>
<tr>
<th>Rule</th>
<th>Premise</th>
<th>Conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>HT-RET</strong></td>
<td>${ True } w { v. v = w }$</td>
<td>$P \Rightarrow P'$</td>
</tr>
<tr>
<td><strong>HT-BIND</strong></td>
<td>$K$ is a context</td>
<td>${ P } e { v. Q } \Rightarrow \forall v. (Q) K(v) { w. R }$</td>
</tr>
<tr>
<td><strong>HT-CSQ</strong></td>
<td>$P \Rightarrow P'$</td>
<td>${ P } e { v. Q }$</td>
</tr>
<tr>
<td><strong>HT-FRAME</strong></td>
<td>${ P } e { v. Q }$</td>
<td>${ P } e { v. Q + R }$</td>
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9.3 Invariant Namespaces

In §9.1, we defined a proposition $P^i$ expressing knowledge (i.e., the proposition is persistent) that $P$ is maintained as invariant with name $i$. The concrete name $i$ is picked when the invariant is allocated, so it cannot possibly be statically known – it will always be a variable that’s threaded through everything. However, we hardly care about the actual, concrete name. All we need to know is that this name is different from the names of other invariants that we want to open at the same time. Keeping track of the $n^2$ mutual inequalities that arise with $n$ invariants quickly gets in the way of the actual proof.

To solve this issue, instead of remembering the exact name picked for an invariant, we will keep track of the namespace the invariant was allocated in. Namespaces are sets of invariants, following a tree-like structure: Think of the name of an invariant as a sequence of identifiers, much like a fully qualified Java class name. A namespace $\mathcal{N}$ then is like a Java package: it is a sequence of identifiers that we think of as containing all invariant names that begin with this sequence. For example, org.mpi-sws.iris is a namespace containing the invariant name org.mpi-sws.iris.heap.

The crux is that all namespaces contain infinitely many invariants, and hence we can freely pick the namespace an invariant is allocated in – no further, unpredictable choice has to be made. Furthermore, we will often know that namespaces are disjoint just by looking at them. The namespaces $\mathcal{N}.\text{iris}$ and $\mathcal{N}.\text{gps}$ are disjoint no matter the choice of $\mathcal{N}$. As a result, there is often no need to track disjointness of namespaces, we just have to pick the namespaces that we allocate our invariants in accordingly.

Formally speaking, let $\mathcal{N} \in \text{InvNamesp} \triangleq \text{List}(\mathcal{N})$ be the type of invariant namespaces. We use the notation $\mathcal{N}.i$ for the namespace $[i] + \mathcal{N}$. (In other words, the list is “backwards”: This is because cons-ing to the list, like the dot does above, is easier to deal with in Coq than appending at the end.)

The elements of a namespaces are structured invariant names (think: Java fully qualified class name). They, too, are lists of $\mathcal{N}$, the same type as namespaces. In order to connect this up to the definitions of §9.1, we need a way to map structured invariant names to InvName, the type of “plain” invariant names. Any injective mapping namesp_inj will do; and such a mapping has to exist because List($\mathcal{N}$) is countable and InvName is infinite. Whenever needed, we (usually implicitly) coerce $\mathcal{N}$ to its encoded suffix-closure, i.e., to the set of encoded structured invariant names contained in the namespace:

$$\mathcal{N}^+ \triangleq \{i | \exists \mathcal{N}', i = \text{namesp_inj}((\mathcal{N}' + \mathcal{N}))\}$$

We will overload the notation for invariant propositions for using namespaces instead of names:

$$P^\mathcal{N} \triangleq \exists i \in \mathcal{N}^+. 
\{P^i\}$$
We can now derive the following rules (this involves unfolding the definition of fancy updates):

\[
\begin{align*}
\text{INV-PERSIST} & : P^N \models \square P^N \\
\text{INV-ALLOC} & : P \models \implies_0 P^N \\
\text{INV-OPEN} & : N \subseteq \mathcal{E} \\
\text{INV-OPEN-TIMELLESS} & : \mathcal{E} \subseteq \mathcal{N}, \text{timeless}(P) 
\end{align*}
\]

9.4 Accessors

The two rules \textsc{inv-open} and \textsc{inv-open-timeless} above may look a little surprising, in the sense that it is not clear on first sight how they would be applied. The rules are the first accessors that show up in this document. Accessors are propositions of the form

\[
P \implies_{\mathcal{E}_1} \exists x. Q \equiv_{-\mathcal{E}_2} \exists x. R
\]

One way to think about such propositions is as follows: Given some accessor, if during our verification we have the proposition \(P\) and the mask \(\mathcal{E}_1\) available, we can use the accessor to access \(Q\) and obtain the witness \(x\). We call this \textit{opening} the accessor, and it changes the mask to \(\mathcal{E}_2\). Additionally, opening the accessor provides us with \(\forall y. Q' \equiv_{\mathcal{E}_2} \exists x. R\), a \textit{linear view shift} (i.e., a view shift that can only be used once). This linear view shift tells us that in order to close the accessor again and go back to mask \(\mathcal{E}_1\), we have to pick some \(y\) and establish the corresponding \(Q'\). After closing, we will obtain \(R\).

Using \textsc{vs-trans} and \textsc{ht-atomic} (or the corresponding proof rules for fancy updates and weakest preconditions), we can show that it is possible to open an accessor around any view shift and any \textit{atomic} expression:

\[
\begin{align*}
\textsc{Acc-vs} & : P \implies_{\mathcal{E}_1} \exists x. Q' \equiv_{\mathcal{E}_2} \exists x. R \quad \forall x. Q \equiv_{\mathcal{E}_1} P \equiv_{\mathcal{E}_2} \exists x. Q \equiv_{\mathcal{E}_1} P \\
\textsc{Acc-HT} & : P \implies_{\mathcal{E}_1} \exists x. Q' \equiv_{\mathcal{E}_2} \exists x. R \quad \forall x. \{Q \equiv_{\mathcal{E}_1} P\} \equiv_{\mathcal{E}_2} \exists x. \{Q' \equiv_{\mathcal{E}_1} P\} \equiv_{\mathcal{E}_1} \text{atomic}(e)
\end{align*}
\]

Furthermore, in the special case that \(\mathcal{E}_1 = \mathcal{E}_2\), the accessor can be opened around any expression. For this reason, we also call such accessors \textit{non-atomic}.

The reasons accessors are useful is that they let us talk about “opening X” (e.g., “opening invariants”) without having to care what X is opened around. Furthermore, as we construct more sophisticated and more interesting things that can be opened (e.g., invariants that can be “cancelled”, or STSs), accessors become a useful interface that allows us to mix and match different abstractions in arbitrary ways.

For the special case that \(P = R\) and \(Q = Q'\), we use the following notation that avoids repetition:

\[
(P \iff x. Q')_{\mathcal{E}_1} \equiv P \implies_{\mathcal{E}_1} \exists x. Q \iff_{\mathcal{E}_1} P
\]

This accessor is “idempotent” in the sense that it does not actually change the state. After applying it, we get our \(P\) back so we end up where we started.
10 Derived Constructions

10.1 Cancellable Invariants

Iris invariants as described in §9.1 are persistent—once established, they hold forever. However, based on them, it is possible to encode a form of invariants that can be “cancelled” again.

First, we need some ghost state:

\[ \text{CInvTok} \triangleq \text{Frac} \]

Now we define:

\[
[\text{CInv} : \gamma]_q \triangleq \left[ \frac{\gamma}{q} \right] \\
\text{CInv}^{\gamma - \mathcal{N}}(P) \triangleq \left[ P \vee \frac{\gamma}{1} \right]^{\mathcal{N}}
\]

It is then straightforward to prove:

\[
\begin{align*}
\text{CInv-new} & : \triangleright P \implies \exists \gamma. [\text{CInv} : \gamma]_1 \star \square \text{CInv}^{\gamma - \mathcal{N}}(P) \\
\text{CInv-acc} & : \triangleright \{ [\text{CInv} : \gamma]_q \iff \triangleright P \}_N \\
\text{CInv-cancel} & : \triangleright \{ \text{CInv}^{\gamma - \mathcal{N}}(P) \} \vdash [\text{CInv} : \gamma]_1 \Rightarrow \mathcal{N} \triangleright P
\end{align*}
\]

Cancellable invariants are useful, for example, when reasoning about data structures that will be deallocated: Every reference to the data structure comes with a fraction of the token, and when all fractions have been gathered, \text{CInv-cancel} is used to cancel the invariant, after which the data structure can be deallocated.

10.2 Non-atomic (“Thread-Local”) Invariants

Sometimes it is necessary to maintain invariants that we need to open non-atomically. Clearly, for this mechanism to be sound we need something that prevents us from opening the same invariant twice, something like the masks that avoid reentrancy on the “normal”, atomic invariants. The idea is to use tokens\(^6\) that guard access to non-atomic invariants. Having the token \([\text{NaInv} : p.\mathcal{E}]\) indicates that we can open all invariants in \(\mathcal{E}\). The \(p\) here is the name of the invariant pool. This mechanism allows us to have multiple, independent pools of invariants that all have their own namespaces.

One way to think about non-atomic invariants is as “thread-local invariants”, where every pool is a thread. Every thread thus has its own, independent set of invariants. Every thread threads through all the tokens for its own pool, so that each invariant can only be opened in the thread it belongs to. As a consequence, they can be kept open around any sequence of expressions (i.e., there is no restriction to atomic expressions) – after all, there cannot be any races with other threads.

Concretely, this is the monoid structure we need:

\[
\begin{align*}
\text{PId} & \triangleq \text{GName} \\
\text{NaTok} & \triangleq \mathcal{V}^{\text{fin}}(\text{InvName}) \times \mathcal{V}(\text{InvName})
\end{align*}
\]

\(^6\)Very much like the tokens that are used to encode “normal”, atomic invariants
For every pool, there is a set of tokens designating which invariants are enabled (closed). This corresponds to the mask of “normal” invariants. We re-use the structure given by namespaces for non-atomic invariants. Furthermore, there is a finite set of invariants that is disabled (open).

Owning tokens is defined as follows:

\[
\begin{align*}
[\text{NaInv} : p.E] & \triangleq \left(\emptyset, E\right) \\
[\text{NaInv} : p] & \triangleq [\text{NaInv} : p.\top]
\end{align*}
\]

Next, we define non-atomic invariants. To simplify this construction, we piggy-back into “normal” invariants.

\[
\begin{align*}
\text{NaInv}^p_N(P) & \triangleq \exists i \in N.\left( P * \{i\} \right)^p \lor [\text{NaInv} : p.\{i\}]^N \\
\end{align*}
\]

We easily obtain:

- **NAInv-new-pool**
  \[\text{True} \Rightarrow \bot \Rightarrow \exists p. [\text{NaInv} : p]\]
- **NAInv-tok-split**
  \[[\text{NaInv} : p.E_1 \uplus E_2] \Leftrightarrow [\text{NaInv} : p.E_1] * [\text{NaInv} : p.E_2]\]
- **NAInv-new-inv**
  \[\vdash P \Rightarrow N \sqcup \text{NaInv}^p_N(P)\]
- **NAInv-acc**
  \[\vdash N \sqsubset E\]

from which we can derive

\[
\begin{align*}
\text{NaInv}^p_N(P) \vdash [[\text{NaInv} : p.E] \Leftrightarrow \vdash P \uplus [\text{NaInv} : p.E \setminus N]]^N
\end{align*}
\]

### 10.3 Boxes

The idea behind the boxes is to have a proposition \(P\) that is actually split into a number of pieces, each of which can be taken out and back in separately. In some sense, this is a replacement for having an “authoritative PCM of Iris propositions itself”. It is similar to the pattern involving saved propositions that was used for the barrier [5], but more complicated because there are some operations that we want to perform without a later.

Roughly, the idea is that a box is a container for an proposition \(P\). A box consists of a bunch of slices which decompose \(P\) into a separating conjunction of the propositions \(Q_i\) governed by the individual slices. Each slice is either full (it right now contains \(Q_i\)), or empty (it does not contain anything currently). The proposition governing the box keeps track of the state of all the slices that make up the box. The crux is that opening and closing of a slice can be done even if we only have ownership of the boxes “later” (\(\triangleright\)).

The interface for boxes is as follows: The two core propositions are: \(\text{BoxSlice}(N, P, i)\), saying that there is a slice in namespace \(N\) with name \(i\) and content \(P\); and \(\text{Box}(N, P, f)\), saying that \(f\) describes the slices of a box in namespace \(N\), such that all the slices together contain \(P\). Here, \(f\) is
of type $\mathbb{N} \xrightarrow{\text{fin}} \text{BoxState}$ mapping names to states, where $\text{BoxState} \triangleq \{ \text{full}, \text{empty} \}$.

$$
\text{Box-create} \quad \begin{array}{c}
\text{True} \Rightarrow_{\mathcal{N}} \text{Box}({\mathcal{N}}, \text{True}, \emptyset)
\end{array}
$$

**Slice-insert-empty**

$\triangleright^b \text{Box}({\mathcal{N}}, P, f) \Rightarrow_{\mathcal{N}} \exists i \notin \text{dom}(f). \Box \text{Slice}({\mathcal{N}}, Q, i) \triangleright^b \text{Box}({\mathcal{N}}, P \ast Q, f[i \leftarrow \text{empty}])$

**Slice-delete-empty**

$f(i) = \text{empty}
\text{BoxSlice}({\mathcal{N}}, Q, i) \triangleright^b \triangleright^b (P = P' \ast Q) \ast \text{Box}({\mathcal{N}}, P', f[i \leftarrow \bot])$

**Slice-fill**

$f(i) = \text{empty}
\text{BoxSlice}({\mathcal{N}}, Q, i) \triangleright^b \triangleright^b Q \ast \triangleright^b \text{Box}({\mathcal{N}}, P, f) \Rightarrow_{\mathcal{N}} \exists i / \in \text{dom}(f).
\Box \text{SliceInv}(i, P)
$

**Slice-empty**

$f(i) = \text{full}
\text{BoxSlice}({\mathcal{N}}, Q, i) \triangleright^b \triangleright^b \text{Box}({\mathcal{N}}, P, f) \Rightarrow_{\mathcal{N}} \exists i / \in \text{dom}(f).
\Box \text{SliceInv}(i, P)
$

Above, $\triangleright^b P$ is syntactic sugar for $\triangleright P$ (if $b$ is 1) or $P$ (if $b$ is 0). This is essentially an optional later, indicating that the lemmas can be applied with Box being owned now or later, and that ownership is returned the same way.

**Model.** The above rules are validated by the following model. We need a camera as follows:

$$
\text{BoxState} \triangleq \text{full} + \text{empty}
\text{Box} \triangleq \text{AUTH}((\text{BoxState})^\gamma) \times \text{AG}(\triangleright i\text{Prop})^\gamma
$$

Now we can define the propositions:

$$
\text{SliceInv}(i, P) \triangleq \exists b \left[ (\triangleright b, \varepsilon)^i \ast ((b = \text{full}) \Rightarrow P) \right]
\text{BoxSlice}({\mathcal{N}}, P, i) \triangleq \left[ (\triangleright \varepsilon, \triangleright P)^i \ast \text{SliceInv}(i, P) \right]^{{\mathcal{N}}}
\text{Box}({\mathcal{N}}, P, f) \triangleq \exists Q : \mathbb{N} \rightarrow i\text{Prop}. \triangleright (P = \bigast_{i \in \text{dom}(f)} Q(i)) \ast (\bigast_{i \in \text{dom}(f)} i\triangleright f[i], Q[i], \text{SliceInv}(i, Q(i)))^{{\mathcal{N}}}
$$
Derived rules. Here are some derived rules:

**Slice-insert-full**

\[ \triangleright Q \triangleright b \text{Box}(N, P, f) \Rightarrow \exists i \notin \text{dom}(f). \Box \text{BoxSlice}(N, Q, i) \triangleright b \text{Box}(N, P \ast Q, f[i \leftarrow \text{full}]) \]

**Slice-delete-full**

\[ f(i) = \text{full} \]

\[ \Box \text{BoxSlice}(N, Q, i) \triangleright b \text{Box}(N, P, f) \Rightarrow \exists \exists P'. \triangleright b(P = P' \ast Q) \ast \Box(N, P', f[i \leftarrow \bot]) \]

**Slice-split**

\[ f(i) = s \]

\[ \Box \text{BoxSlice}(N, Q_1 \ast Q_2, i) \triangleright b \text{Box}(N, P, f) \Rightarrow \exists \exists i_1 \notin \text{dom}(f), i_2 \notin \text{dom}(f), i_1 \neq i_2 \land \Box \text{BoxSlice}(N, Q_1, i_1) \ast \Box \text{BoxSlice}(N, Q_2, i_2) \ast \triangleright b \Box(N, P, f[i \leftarrow \bot][i_1 \leftarrow s][i_2 \leftarrow s]) \]

**Slice-merge**

\[ i_1 \neq i_2 \quad f(i_1) = f(i_2) = s \]

\[ \Box \text{BoxSlice}(N, Q_1, i_1), \Box \text{BoxSlice}(N, Q_2, i_2) \triangleright b \text{Box}(N, P, f) \Rightarrow \exists \exists i \notin \text{dom}(f) \setminus \{i_1, i_2\}. \Box \text{BoxSlice}(N, Q_1 \ast Q_2, i) \ast \triangleright b \Box(N, P, f[i_1 \leftarrow \bot][i_2 \leftarrow \bot][i \leftarrow s]) \]
11 Logical Paradoxes

In this section we provide proofs of some logical inconsistencies that arise when slight changes are made to the Iris logic.

11.1 Saved Propositions without a Later

As a preparation for the proof about invariants in §11.2, we show that omitting the later modality from a variant of saved propositions leads to a contradiction. Saved propositions have been introduced in prior work [4, 5] to prove correctness of synchronization primitives; we will explain all that is necessary here. The counterexample assumes a higher-order logic with separating conjunction, magic wand and the modalities □ and ⇒ satisfying the rules in §5.

Theorem 4. If there exists a type GName and a proposition _ ⊨ _ : GName → iProp → iProp associating names γ : GName to propositions and satisfying:

\[ \vdash _ \Rightarrow \exists \gamma : GName. \gamma \Rightarrow P(\gamma) \]  \hspace{1cm} (sprop-alloc)
\[ \gamma \Rightarrow P \vdash \Box(\gamma \Rightarrow P) \]  \hspace{1cm} (sprop-persist)
\[ \gamma \Rightarrow P \ast \gamma \Rightarrow Q \vdash P \iff Q \] \hspace{1cm} (sprop-agree)

then \[ \vdash _ \Rightarrow \text{False} \].

The type GName should be thought of as the type of “locations” and γ ⇒ P should be read as stating that location γ “stores” proposition P. Notice that these are immutable locations, so the maps-to proposition is persistent. The rule sprop-alloc is then thought of as allocation, and the rule sprop-agree states that a given location γ can only store one proposition, so multiple witnesses covering the same location must agree.

The conclusion of sprop-agree usually is guarded by a □. The point of this theorem is to show that said later is essential, as removing it introduces inconsistency. The key to proving Theorem 4 is the following proposition:

Definition 22. A(γ) ≜ ∃P : iProp. □¬P ∧ γ ⇒ P.

Intuitively, A(γ) says that the saved proposition named γ does not hold, i.e., we can disprove it. Using sprop-persist, it is immediate that A(γ) is persistent.

Now, by applying sprop-alloc with A, we obtain a proof of P ≜ γ ⇒ A(γ): this says that the proposition named γ is the proposition saying that it, itself, does not hold. In other words, P says that the proposition named γ expresses its own negation. Unsurprisingly, that leads to a contradiction, as is shown in the following lemma:

Lemma 2. We have γ ⇒ A(γ) ⊨ □¬A(γ) and γ ⇒ A(γ) ⊨ A(γ).

Proof.

- First we show γ ⇒ A(γ) ⊨ □¬A(γ). Since γ ⇒ A(γ) is persistent it suffices to show γ ⇒ A(γ) ⊨ ¬A(γ). Suppose γ ⇒ A(γ) and A(γ). Then by definition of A there is a P such that □¬P and γ ⇒ P. By sprop-agree we have P ⇒ A(γ) and so from ¬P we get ¬A(γ), which leads to a contradiction with A(γ).
• Using the first item we can now prove \( \gamma \implies A(\gamma) \implies A(\gamma) \). We need to prove

\[
\exists P : iProp. \quad \Box \neg P \land \gamma \implies P.
\]

We do so by picking \( P \) to be \( A(\gamma) \), which leaves us to prove \( \Box \neg A(\gamma) \land \gamma \implies A(\gamma) \). The last conjunct holds by assumption, and the first conjunct follows from the previous item of this lemma.

With this lemma in hand, the proof of Theorem 4 is simple.

\textit{Theorem 4.} Using the previous lemmas we have

\[
\vdash \forall \gamma. \neg (\gamma \implies A(\gamma)).
\]

Together with the rule \textsc{sprop-alloc} we thus derive \( \models False \).

\[\square\]

11.2 Invariants without a Later

Now we come to the main paradox: if we remove the \( \triangleright \) from \textsc{inv-open}, the logic becomes inconsistent. The theorem is stated as general as possible so that it also applies to previous, less powerful versions of Iris.

\textit{Theorem 5.} Assume a higher-order separation logic with \( \Box \) and an update modality with a binary mask \( \{0, 1\} \) (think: empty mask and full mask) satisfying strong monad rules with respect to separating conjunction and such that:

\[
\text{WEAKEN-MASK} \quad \models 0 P \models 1 P
\]

Assume a type \textsc{InvName} and a proposition \( \square : \textsc{InvName} \rightarrow iProp \rightarrow iProp \) satisfying:

\[
\text{INV-ALLOC} \quad P \models \exists \iota. P
\]

\[
\text{INV-PERSIST} \quad P \models \Box P
\]

\[
\text{INV-OPEN-NOLATER} \quad P * Q \models 0 (P * R)
\]

Finally, assume the existence of a type \textsc{GName} and two tokens \( s : \textsc{GName} \rightarrow iProp \) and \( f : \textsc{GName} \rightarrow iProp \) parameterized by \textsc{GName} and satisfying the following properties:

\[
\text{START-ALLOC} \quad \models 0 \gamma. s \gamma
\]

\[
\text{START-FINISH} \quad s \gamma \models 0 f \gamma
\]

\[
\text{START-NOT-FINISHED} \quad s \gamma * f \gamma \models False
\]

\[
\text{FINISHED-DUP} \quad f \gamma \models 0 f \gamma * f \gamma
\]

Then \( \text{True} \models 1 False \).

The core of the proof is defining the \( \models \) from the previous counterexample using invariants. Then, using the standard proof rules for invariants, we show that it satisfies \textsc{sprop-alloc} and \textsc{sprop-persist}. Furthermore, assuming the rule for opening invariants without a \( \triangleright \), we can prove a slightly weaker version of \textsc{sprop-agree}, which is sufficient for deriving a contradiction.

We start by defining \( \models \) satisfying (almost) the assumptions of Lemma 4.
Definition 23. We define $\_ \Rightarrow \_ : GName \to iProp \to iProp$ as:

$$\gamma \Rightarrow P \triangleq \exists \iota. [\langle [S_{\gamma}]^{\gamma} \lor [F_{\gamma}]^{\gamma} * \Box P \rangle].$$

Note that using INV-PERSIST, it is immediate that $\gamma \Rightarrow P$ is persistent.

We use the tokens $[S_{\gamma}]^{\gamma}$ and $[F_{\gamma}]^{\gamma}$ to model invariants that can be initialized “lazily”: $[S_{\gamma}]^{\gamma}$ indicates that the invariant is still not initialized, whereas the duplicable $[F_{\gamma}]^{\gamma}$ indicates it has been initialized with a resource satisfying $P$.

We can show variants of SPROP-AGREE and SPROP-ALLOC for the defined $\Rightarrow$.

Lemma 3. We have $\vdash \Rightarrow_{1} \exists \gamma. \gamma \Rightarrow P(\gamma)$.

Proof. We have to show the allocation rule

$$\vdash \Rightarrow_{1} \exists \gamma. \gamma \Rightarrow P.$$

From START-ALLOC we have a $\gamma$ such that $\Rightarrow_{0} \langle [S_{\gamma}]^{\gamma} \rangle$ holds and hence from WEAKEN-MASK we have $\Rightarrow_{1} \langle [S_{\gamma}]^{\gamma} \rangle$. Since we are proving a goal of the form $\Rightarrow_{1} R$ we may assume $[S_{\gamma}]^{\gamma}$. Thus for any $P$ we have $\Rightarrow_{1} \langle [S_{\gamma}]^{\gamma} \lor [F_{\gamma}]^{\gamma} * P \rangle$. Again since our goal is still of the form $\Rightarrow_{1} R$ we may assume $[S_{\gamma}]^{\gamma} \lor [F_{\gamma}]^{\gamma} * \Box P$.

The rule INV-ALLOC then gives us precisely what we need.

Lemma 4. We have $\gamma \Rightarrow P * \gamma \Rightarrow Q * \Box P \vdash \Rightarrow_{1} \Box Q$ and thus $\gamma \Rightarrow P * \gamma \Rightarrow Q \vdash (\Rightarrow_{1} \Box P) \leftrightarrow (\Rightarrow_{1} \Box Q)$.

Lemma 4. We first show $\gamma \Rightarrow P * \gamma \Rightarrow Q * \Box P \vdash \Rightarrow_{1} \Box Q$.

We use INV-OPEN-NOLATER to open the invariant in $\gamma \Rightarrow P$ and consider two cases:

1. $[S_{\gamma}]^{\gamma}$ (the invariant is “uninitialized”): In this case, we use START-FINISH to “initialize” the invariant and obtain $[F_{\gamma}]^{\gamma}$. Then we duplicate $[F_{\gamma}]^{\gamma}$, and use it together with $\Box P$ to close the invariant.

2. $[F_{\gamma}]^{\gamma} * \Box P$ (the invariant is “initialized”): In this case we duplicate $[F_{\gamma}]^{\gamma}$, and use a copy to close the invariant.

After closing the invariant, we have obtained $[F_{\gamma}]^{\gamma}$. Hence, it is sufficient to prove

$$[F_{\gamma}]^{\gamma} * \gamma \Rightarrow P * \gamma \Rightarrow Q * \Box P \vdash \Rightarrow_{1} \Box Q.$$

We proceed by using INV-OPEN-NOLATER to open the other invariant in $\gamma \Rightarrow Q$, and we again consider two cases:

1. $[S_{\gamma}]^{\gamma}$ (the invariant is “uninitialized”): As witnessed by START-NOT-FINISHED, this cannot happen, so we derive a contradiction. Notice that this is a key point of the proof: because the two invariants ($\gamma \Rightarrow P$ and $\gamma \Rightarrow Q$) share the ghost name $\gamma$, initializing one of them is enough to show that the other one has been initialized. Essentially, this is an indirect way of saying that really, we have been opening the same invariant two times.

2. $[F_{\gamma}]^{\gamma} * \Box Q$ (the invariant is “initialized”): Since $\Box Q$ is duplicable we use one copy to close the invariant, and retain another to prove $\Rightarrow_{1} \Box Q$. 

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• By applying the above twice, we easily obtain

\[ \gamma \Rightarrow P \Rightarrow Q \vdash (\triangleright \square P) \iff (\triangleright \square Q) \]

\[ \square \quad \quad \square \]

Intuitively, Lemma 4 shows that we can “convert” a proof from \( P \) to \( Q \).

We are now in a position to replay the counterexample from §11.1. The only difference is that because Lemma 4 is slightly weaker than the rule sprop-agree of Theorem 4, we need to use \( \triangleright \square \text{False} \) in place of \( \text{False} \) in the definition of the predicate \( A \): we let \( A(\gamma) \triangleq \exists P : \text{iProp. } \square(P \Rightarrow \triangleright \square \text{False}) \land \gamma \Rightarrow P \) and replay the proof that we have presented above.
References


